Surmise Relations between Tests - Preliminary Results of the Mathematical Modeling ⋆

Silke Brandt, Dietrich Albert, and Cord Hockemeyer

Department of Psychology, Karl-Franzens-Universität Graz, 8010 Graz, Austria

Abstract

Our work is based upon the theory of knowledge spaces, which was introduced by Doignon and Falmagne in 1985. They used prerequisite relationships between items within a body of information for the assessment and training of knowledge. Often it is useful to partition such a body of information into special fields - subjects, tests, courses or whatever. As we are mostly interested in psychological applications, we will refer to these special fields as *tests*, but, generalized, it is of course also possible to regard subjects, courses or grades instead of tests.

Regarding a given set of such tests, we now want to investigate the relations and dependencies, parallelity and unifiability of these tests. Therefore, we extend the concept of prerequisite relationships between items *within* tests to prerequisite relationships *between* tests. Such a prerequisite relationship is the *surmise relation between tests*. After its definition we discuss its properties and special cases. Then we introduce the *test knowledge space*, which plays a central role in our concept. Under certain circumstances it is possible to find a *base* for a test knowledge space. The base is a very efficient way of storing information about the test knowledge space. Moreover, we show that by means of the base not only the test knowledge space, but also the surmise relation between tests and its properties can be inferred.

As this is a report on research in progress, we finally want to give a short overview about the further research regarding this mathematical model. This model will be a basis for a software system that will analyze tests as well as partition sets of items into tests.

Key words: Surmise relations between tests. Left-covering surmise relation. Right-covering surmise relation. Test knowledge spaces.

⋆ Expanded version of a talk presented at the Meeting on Ordinal and Symbolic Data Analysis (Amherst, MA, September 1998). This work is supported by the Austrian Science Fund (FWF) through the project grant P12726-SOZ to Dietrich Albert and Wilhelm Schappacher.

1 Introduction

First some basic definitions of the knowledge space theory will be explained. These concepts have been introduced by Doignon & Falmagne[3,5,1,4].

Definition 1 *A* ***knowledge structure*** *is a pair* $(Q, \mathcal{K})$ *in which* $Q \neq \emptyset, \mathcal{K} \subseteq 2^Q, \emptyset \in \mathcal{K}$ *and* $Q \in \mathcal{K}$*. The set* Q *is called the* ***domain*** *of the knowledge structure and its elements are called* ***items****. We also say that* $\mathcal{K}$ *is a knowledge structure on a set* Q*. The elements of* $\mathcal{K}$ *are called* ***knowledge states****.*

In our psychological interpretation we consider Q as a set of problems respectively questions (e.g. a test in arithmetics). The knowledge state of a person is then the set of all problems that this person is capable of solving. The knowledge structure $\mathcal{K}$ is the collection of all occuring knowledge states.

Definition 2 *A knowledge structure* $(Q, \mathcal{K})$ *is called a* ***knowledge space*** *iff* $\mathcal{K}$ *is closed under union*[1]*. A knowledge space* $(Q, \mathcal{K})$ *is called* ***quasi ordinal*** *iff* $\mathcal{K}$ *is closed under intersection*[2]*.*

Let $(Q, \mathcal{K})$ be a knowledge structure, $x \in Q$. Then $\mathcal{K}_{\mathbf{x}}$ denotes the collection of all knowledge states containing the item x:

$$\mathcal{K}_x := \{K \in \mathcal{K} \mid x \in K\}.$$

Now we can define a prerequisite relation on the set Q.

Definition 3 *The relation* $S \subseteq Q \times Q$ *defined by*

$$ySx \Leftrightarrow y \in \bigcap \mathcal{K}_x{}^3 \quad \forall x, y \in Q$$

is called the ***surmise relation*** *of the knowledge structure. When* ySx *holds, we say that* y *is* ***surmisable*** *from* x*.*

ySx holds iff y is an element of all the knowledge states which contain the item x. Thus, for our interpretation, each person, who masters problem x, also masters problem y. y is a prerequisite for x. Thus, from the performance of problem x we can surmise the performance of problem y (see Figure 1).

---

[1] $\mathcal{K}$ is closed under union iff $K_1 \cup K_2 \in \mathcal{K}$ holds $\forall K_1, K_2 \in \mathcal{K}$

[2] $\mathcal{K}$ is closed under intersection iff $K_1 \cap K_2 \in \mathcal{K}$ holds $\forall K_1, K_2 \in \mathcal{K}$

[3] $\bigcap \mathcal{K}_x := \bigcap\limits_{K \in \mathcal{K}_x} K$

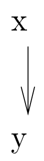

Fig. 1. y is a prerequisite of x

2 Surmise Relations between Tests

Till now we regarded single items and surmise relations between these items *within* a set Q. Now we will consider a partition of this set Q into tests $A, B, C, \ldots$, where $A, B, C, \ldots \neq \emptyset$ and pairwise disjoint. In the following let $\mathcal{T} = \{A, B, C, \ldots\}$ denote the whole set of these tests. For $x \in Q$ and $B \in \mathcal{T}$ let $B_x := B \cap \bigcap \mathcal{K}_x$.

Definition 4 *The relation $\dot{\mathcal{S}} \subseteq \mathcal{T} \times \mathcal{T}$ defined by*

$$B \; \dot{\mathcal{S}} \; A \Leftrightarrow \; \exists a \in A : B_a \neq \emptyset \qquad \forall A, B \in \mathcal{T}$$

is called ***surmise relation between tests****. When $B \; \dot{\mathcal{S}} \; A$ holds we say A and B are in* ***surmise relation from*** *A* ***to*** *B or shorter: the pair (B, A) is in surmise relation.*

Surmise relations between tests are interpreted in the following way: For a given item or set of items in test A a person is able to perform, we can surmise at least the performance of a nonempty subset of test B (see Figure 2). The idea of surmise relations between tests was introduced by Albert[2].

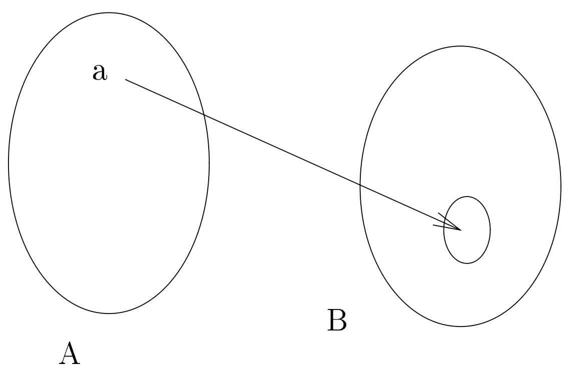

Fig. 2. A and B are in surmise relation from A to B

Now we want to investigate the properties of surmise relations between tests. The question occurs whether it is possible to transfer the properties of surmise relations between items to surmise relations between tests. We know that the surmise relation between items is a quasi order, that is it is reflexive[4] and transitive[5].

Corollary 5 *The surmise relation between tests is reflexive, as well.*

PROOF.

$$\forall a \in A : \exists K \in \mathcal{K} \text{ with } a \in K (\text{as } Q \in \mathcal{K} \wedge A \in Q) \Rightarrow$$
$$\forall a \in A : a \in \bigcap \mathcal{K}_a \Rightarrow$$
$$\forall a \in A : a \in A \cap \bigcap \mathcal{K}_a = A_a \Rightarrow A \, \dot{\mathcal{S}} \, A.$$

□

See Figure 3.

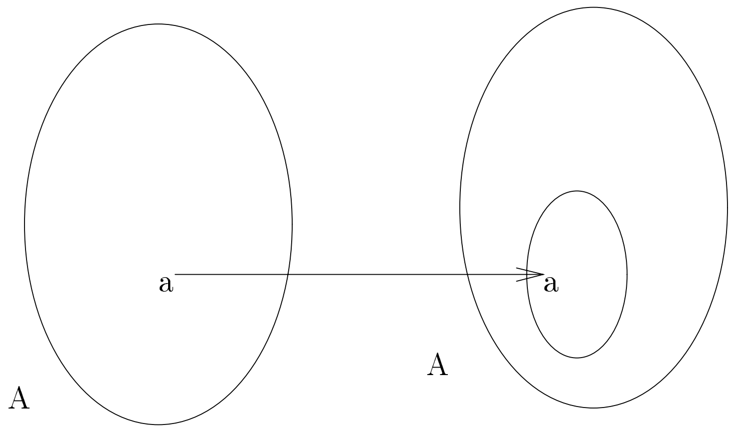

Fig. 3. $A \, \dot{\mathcal{S}} \, A$

Corollary 6 *The surmise relation between tests is not necessarily transitive.*

PROOF.

Let A, B, C be tests and suppose $C \, \dot{\mathcal{S}} \, B$ and $B \, \dot{\mathcal{S}} \, A \Rightarrow$
$$\exists b \in B : C_b \neq \emptyset \wedge \ \exists a \in A : B_a \neq \emptyset \Rightarrow$$
$$\exists b \in B : C \cap \bigcap \mathcal{K}_b \neq \emptyset \wedge \ \exists a \in A : B \cap \bigcap \mathcal{K}_a \neq \emptyset \Rightarrow$$
$$\exists b \in B, \ \exists c \in C : c \in \bigcap \mathcal{K}_b \wedge \ \exists a \in A, \ \exists b_1 \in B : b_1 \in \bigcap \mathcal{K}_a.$$

[4] $aSa \quad \forall a \in Q$

[5] $aSb \wedge bSc \Rightarrow aSc \quad \forall a, b, c \in Q$

If $b = b_1$ then there $\exists a \in A, b \in B, c \in C : b \in \bigcap \mathcal{K}_a \wedge c \in \bigcap \mathcal{K}_b$.
For $b \in \bigcap \mathcal{K}_a \Rightarrow \{K \in \mathcal{K} \mid a \in K\} \subseteq \{K \in \mathcal{K} \mid b \in K\} \Rightarrow \mathcal{K}_a \subseteq \mathcal{K}_b \Rightarrow$ $\bigcap \mathcal{K}_b \subseteq \bigcap \mathcal{K}_a$ and therefore, $c \in \bigcap \mathcal{K}_a$ and $C\ \dot{\mathcal{S}}\ A$.
But if $b \neq b_1$, then no further conclusions are possible (see Figure 4).

□

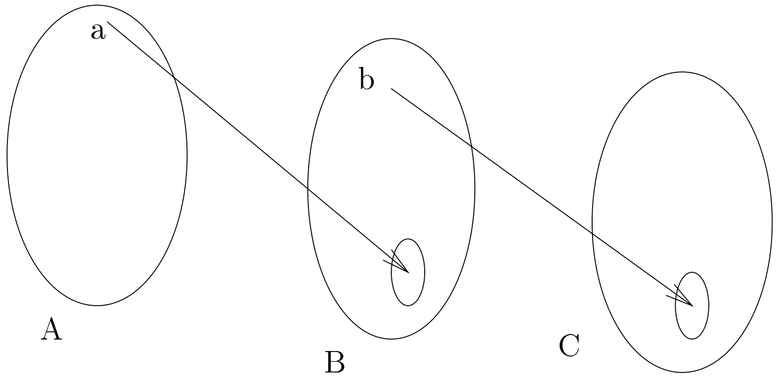

Fig. 4. $C\ \dot{\mathcal{S}}\ B$ and $B\ \dot{\mathcal{S}}\ A$, but $C\ \dot{\not\mathcal{S}}\ A$

Therefore, the surmise relation between tests is not a quasi order. However, there are special cases for which transitivity holds though. The first case occurs, if the surmise relation between tests is *left-covering*.

Definition 7 *A and B are in* ***left-covering surmise relation from A to B*** $\Leftrightarrow \forall a \in A : B_a \neq \emptyset$.
Notation: $B\ \dot{\mathcal{S}}_l\ A$.

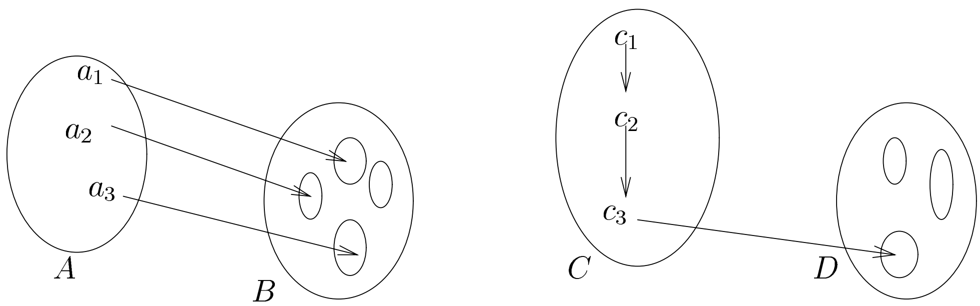

Fig. 5. $B\ \dot{\mathcal{S}}_l\ A$ and $D\ \dot{\mathcal{S}}_l\ C$

That is, from the performance of any item in test A we can at least surmise the performance of a nonempty subset of items in test B.
The surmise relation between tests is called left-covering, iff $\forall A, B \in \mathcal{T}$: $A\ \dot{\mathcal{S}}\ B \Rightarrow A\ \dot{\mathcal{S}}_l\ B$ holds.

Corollary 8 $\dot{\mathcal{S}}_l$ *is reflexive on* $\mathcal{T}$.

PROOF.

$\forall a \in A : A_a \neq \emptyset$ (See proof of Cor.5) $\Rightarrow A \; \dot{\mathcal{S}}_l \; A$.

$\square$

Corollary 9 $\dot{\mathcal{S}}_l$ *is transitive on* $\mathcal{T}$.

PROOF.

Suppose $C \; \dot{\mathcal{S}}_l \; B \wedge B \; \dot{\mathcal{S}}_l \; A \Rightarrow$
$\forall b \in B \;\; \exists c \in C : c \in \bigcap \mathcal{K}_b \wedge \forall a \in A \;\; \exists b \in B : b \in \bigcap \mathcal{K}_a \Rightarrow$
$\forall a \in A \;\; \exists b \in B, c \in C : b \in \bigcap \mathcal{K}_a \wedge c \in \bigcap \mathcal{K}_b \Rightarrow$
$\forall a \in A \;\; \exists c \in C : c \in \bigcap \mathcal{K}_a$ (see proof of Cor.6) $\Rightarrow C \; \dot{\mathcal{S}}_l \; A$.

$\square$

See Figure 6.

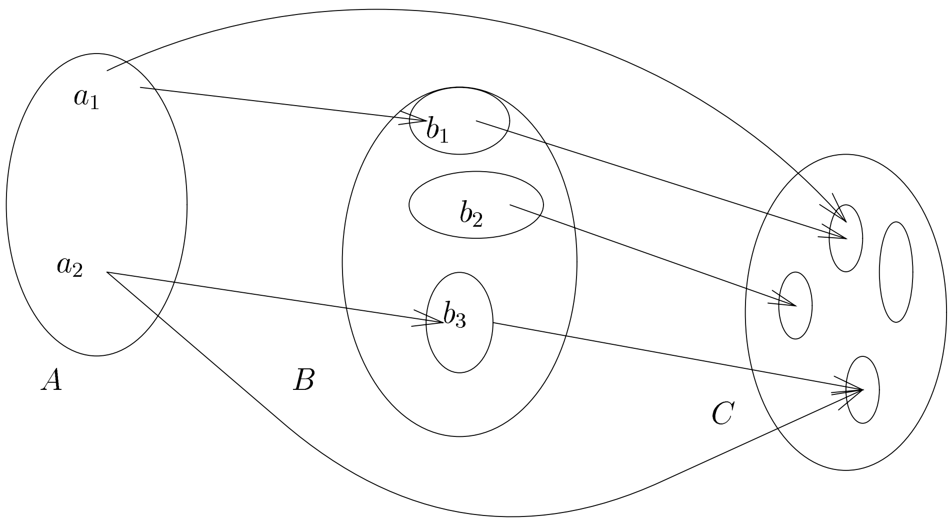

Fig. 6. $C \; \dot{\mathcal{S}}_l \; B \wedge B \; \dot{\mathcal{S}}_l \; A \Rightarrow C \; \dot{\mathcal{S}}_l \; A$

The second special case occurs, if the surmise relation between tests is *right-covering.*

Definition 10 *A and B are in* ***right-covering surmise relation from A to B*** $\Leftrightarrow \bigcup_{a \in A} B_a = B$.
Notation: $B \, \dot{\mathcal{S}}_r \, A$.

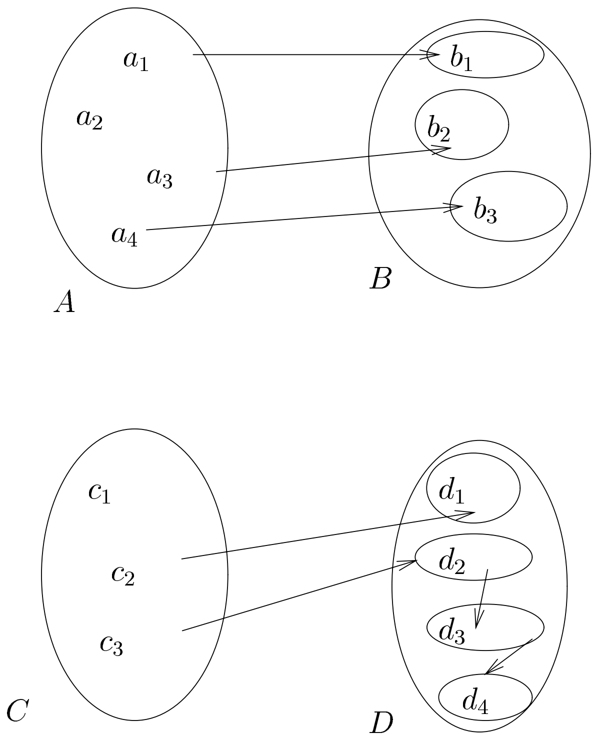

Fig. 7. $A \, \dot{\mathcal{S}}_r \, B$ and $C \, \dot{\mathcal{S}}_r \, D$.

For all items b in test B, there exists an item a in test A for which $b \in B_a$ holds. Thus, for our interpretation, for all items b in test B, exists an item a in test A for which bSa holds. From the performance of the whole test A the performance of the whole test B can be surmised. The whole test B is a prerequisite for the test A.
The surmise relation between tests is called right-covering, iff $\forall A, B \in \mathcal{T}$: $B \, \dot{\mathcal{S}} \, A \Rightarrow B \, \dot{\mathcal{S}}_r \, A$ holds.

Corollary 11 *$\dot{\mathcal{S}}_r$ is reflexive on $\mathcal{T}$.*

PROOF.

$$\forall a_1 \in A : a_1 \in \bigcap \mathcal{K}_{a_1} \Rightarrow \forall a_1 \in A : a_1 \in \bigcup_{a \in A} \left(\bigcap \mathcal{K}_a\right) \Rightarrow$$
$$\bigcup_{a \in A} \left(A \cap \bigcap \mathcal{K}_a\right) = A \Rightarrow \bigcup_{a \in A} A_a = A \Rightarrow A \, \dot{\mathcal{S}} \, A.$$

□

Corollary 12 *$\dot{\mathcal{S}}_r$ is transitive on $\mathcal{T}$.*

PROOF.

$$\begin{aligned}
&\text{Suppose } C\ \dot{\mathcal{S}}_r\ B \wedge B\ \dot{\mathcal{S}}_r\ A \Rightarrow \\
&\forall c \in C\ \ \exists b \in B : c \in \bigcap \mathcal{K}_b \wedge \forall b \in B\ \ \exists a \in A : b \in \bigcap \mathcal{K}_a \Rightarrow \\
&\forall c \in C\ \ \exists a \in A, b \in B : c \in \bigcap \mathcal{K}_b \wedge b \in \bigcap \mathcal{K}_a \Rightarrow \\
&\forall c \in C\ \ \exists a \in A : c \in \bigcap \mathcal{K}_a \Rightarrow \\
&\bigcup_{a \in A} \left(C \cap \bigcap \dot{\mathcal{K}}_a\right) = C \Rightarrow \\
&\bigcup_{a \in A} C_a = C \Rightarrow C\ \dot{\mathcal{S}}_r\ A.
\end{aligned}$$

□

See Figure 8.

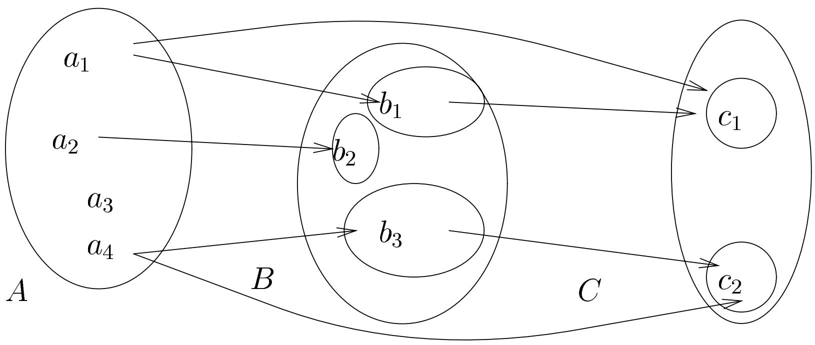

Fig. 8. $C\ \dot{\mathcal{S}}_r\ B \wedge B\ \dot{\mathcal{S}}_r\ A \Rightarrow C\ \dot{\mathcal{S}}_r\ A$.

Both the left-covering and the right-covering surmise relation are quasi orders.

3 Test Knowledge Spaces

The concept of test knowledge spaces is based upon the concept of knowledge spaces.

Definition 13 *For a knowledege state $K_i \in \mathcal{K}$ and $\mathcal{T} = \{A, B, C, \ldots\}$ the n-tupel $\dot{K}_i = (A_i, B_i, \ldots)$, where $A_i = A \cap K_i, B_i = B \cap K_i, \ldots$ for $i \in \mathbf{N}$, is called* ***test knowledge state.*** *Let $\dot{\mathcal{K}}$ denote the collection of all test knowledge states. Then the pair $(\mathcal{T}, \dot{\mathcal{K}})$ is called* ***test knowledge structure.***

In our interpretation, if $\dot{K}_i$ is the test knowledge state of a person, then A_i is the subset of items in test A, which this person is capable of solving, B_i is the subset of items in test B, which this person is capable of solving, and so on.

Definition 14 *A test knowledge structure* $(\mathcal{T}, \dot{\mathcal{K}})$ *is a* ***test knowledge space****, iff* $\dot{\mathcal{K}}$ *is closed under union.* $(\mathcal{T}, \dot{\mathcal{K}})$ *is a* ***quasi ordinal test knowledge space*** *iff* $\dot{\mathcal{K}}$ *is closed under union and intersection.*

Notice that union and intersection for n-tupels is not the same as union and intersection for sets!

Definition 15 *For* $\dot{K}_i = (A_i, B_i, \ldots)$ *and* $\dot{K}_j = (A_j, B_j, \ldots)$*:*
$\dot{K}_i \dot{\cup} \dot{K}_j := (A_i \cup A_j, B_i \cup B_j, \ldots)$.
$\dot{K}_i \dot{\cap} \dot{K}_j := (A_i \cap A_j, B_i \cap B_j, \ldots)$.

Corollary 16 *The test knowledge structure* $\dot{\mathcal{K}}$ *is a test knowledge space* $\Leftrightarrow$ *the corresponding knowledge structure* $\mathcal{K}$ *is a knowledge space. The test knowledge space* $\dot{\mathcal{K}}$ *is quasi ordinal* $\Leftrightarrow$ *the corresponding knowledge space* $\mathcal{K}$ *is quasi ordinal.*

PROOF.

$$\begin{aligned}
&\forall \dot{K}_i = (A_i, B_i, \ldots), \dot{K}_j = (A_j, B_j, \ldots) \in \dot{\mathcal{K}}, K_i, K_j \in \mathcal{K}: \\
&\dot{K}_i \dot{\cup} \dot{K}_j = (A_i \cup A_j, B_i \cup B_j, \ldots) \in \dot{\mathcal{K}} \Leftrightarrow \\
&\exists K_{i\cup j} \in \mathcal{K} : (A_i \cup A_j) = A \cap K_{i\cup j} \Leftrightarrow \\
&\exists K_{i\cup j} \in \mathcal{K} : (A \cap K_i) \cup (A \cap K_j) = A \cap K_{i\cup j} \Leftrightarrow \\
&\exists K_{i\cup j} \in \mathcal{K} : A \cap (K_i \cup K_j) = A \cap K_{i\cup j} \Leftrightarrow \\
&\exists K_{i\cup j} \in \mathcal{K} : K_i \cup K_j = K_{i\cup j} \Leftrightarrow K_i \cup K_j \in \mathcal{K}.
\end{aligned}$$

$$\begin{aligned}
&\forall \dot{K}_i = (A_i, B_i, \ldots), \dot{K}_j = (A_j, B_j, \ldots) \in \dot{\mathcal{K}}, K_i, K_j \in \mathcal{K}: \\
&\dot{K}_i \dot{\cap} \dot{K}_j = (A_i \cap A_j, B_i \cap B_j, \ldots) \in \dot{\mathcal{K}} \Leftrightarrow \\
&K_i \cap K_j \in \mathcal{K}.
\end{aligned}$$

□

4 The Base

Definition 17 *A subcollection* $\mathcal{B} \subseteq \mathcal{K}$ *of states is called* ***base*** *of* $\mathcal{K}$ *iff the following conditions hold:*

(1) All the states of $\mathcal{K}$ can be obtained by taking all arbitrary unions (including the empty union) of the states included in the subcollection $\mathcal{B}$.
$\forall K \subseteq \mathcal{K}\ \exists K_1, \ldots, K_n \in \mathcal{B}, n \in \mathbf{N},$ such that $K = K_1 \cup \ldots \cup K_n$.

(2) $\mathcal{B}$ is minimal in the sense that it is included in any other subcollection of states generating the states in $\mathcal{K}$ by taking unions of states in $\mathcal{B}$.
$\forall \mathcal{P}$ which fulfill 1., holds: $\mathcal{B} \subseteq \mathcal{P}$.

If the set Q of items is finite and the corresponding knowledge structure $\mathcal{K}$ is a knowledge space, it is always possible to find such a base for $\mathcal{K}$. In particular there exists one and only one base for each knowledge space [3,4].

Because of Cor. 16 it is easy to transfer this definition of a base for a knowledge space to the definition of a base for a test knowledge space. We only have to replace $\mathcal{K}$ by $\dot{\mathcal{K}}$, K by $\dot{K}$ and union for sets by the union defined in Def.15.

Definition 18 *$\dot{\mathcal{B}} \subseteq \dot{\mathcal{K}}$ is called **base** of $\dot{\mathcal{K}}$ iff the following conditions hold:*

(1) $\forall \dot{K} \in \dot{\mathcal{K}}\ \exists \dot{K}_1, \ldots, \dot{K}_n \in \dot{\mathcal{B}} : \dot{K} = \dot{K}_1 \dot{\cup} \ldots \dot{\cup} \dot{K}_n$.

(2) $\forall \dot{\mathcal{P}} \subseteq \dot{\mathcal{K}}$ which fulfill 1., holds: $\dot{\mathcal{B}} \subseteq \dot{\mathcal{P}}$.

In particular the following statement holds:

Corollary 19 *Let $(Q, \mathcal{K})$ denote a knowledge structure and $(\mathcal{T}, \dot{\mathcal{K}})$ denote the corresponding test knowledge structure. Then $\dot{\mathcal{B}} = \{(A_i, B_i, C_i, \ldots), (A_j, B_j, C_j, ...), \ldots\}$ is the base of $\dot{\mathcal{K}} \Leftrightarrow \mathcal{B} = \{A_i \cup B_i \cup C_i \ldots, A_j \cup B_j \cup C_j \ldots, \ldots\}$ is the base of $\mathcal{K}$.*

PROOF.

$\mathcal{B} = \{K_1, \ldots, K_n\} \wedge \dot{\mathcal{B}} := \{\dot{K}_1, \ldots, \dot{K}_n\}$ with $\dot{K}_i := (A_i, B_i, C_i, \ldots)$,
$A_i = A \cap K_i, B_i = B \cap K_i, C_i = C \cap K_i, \ldots$ for $i \in \{1, \ldots, n\}$
$\dot{\mathcal{B}} \subseteq \dot{\mathcal{K}} \Leftrightarrow \dot{K}_i \in \dot{\mathcal{K}}$ for $i \in \{1, \ldots, n\} \Leftrightarrow$
$K_i \in \mathcal{K}$ for $i \in \{1, \ldots, n\} \Leftrightarrow \mathcal{B} \subseteq \mathcal{K}$.

$\forall \dot{K}_m := (A_m, B_m, C_m, \ldots) \in \dot{\mathcal{K}}\ \exists \dot{K}_1, \ldots, \dot{K}_j \in \dot{\mathcal{B}}$
with $\dot{K}_m = \dot{K}_1 \dot{\cup} \ldots \dot{\cup} \dot{K}_j \Leftrightarrow$
$\forall \dot{K}_m \in \dot{\mathcal{K}}\ \exists \dot{K}_1, \ldots, \dot{K}_j \in \dot{\mathcal{B}}$ with $A_m = \bigcup_{i=1}^{j} A_i$,
$B_m = \bigcup_{i=1}^{j} B_i,\ C_m = \bigcup_{i=1}^{j} C_i, \ldots \Leftrightarrow$
$\forall K_m \in \mathcal{K}\ \exists K_1, \ldots, K_j \in \mathcal{B}$ with $A_i = A \cap K_i$,

$$B_i = B \cap K_i, \;\; C_i = C \cap K_i \text{ for } i = \{1,..,j\} \wedge$$
$$\bigcup_{i=1}^{j} A_i = A \cap K_m, \bigcup_{i=1}^{j} B_i = B \cap K_m, \ldots \Leftrightarrow$$
$$\forall K_m \in \mathcal{K} \;\; \exists K_1, \ldots, K_j \in \mathcal{B} : \bigcup_{i=1}^{j} (A \cap K_i) = A \cap K_m,$$
$$\bigcup_{i=1}^{j} (B \cap K_i) = B \cap K_m, \ldots \Leftrightarrow$$
$$\forall K_m \in \mathcal{K} \;\; \exists K_1, \ldots, K_j \in \mathcal{B} :$$
$$A \cap \bigcup_{i=1}^{j} K_i = A \cap K_m, B \cap \bigcup_{i=1}^{j} K_i = B \cap K_m, \ldots \Leftrightarrow$$
$$\forall K_m \in \mathcal{K} \;\; \exists K_1, \ldots, K_j \in \mathcal{B} : \bigcup_{i=1}^{j} K_i = K_m.$$
$$\forall \dot{\mathcal{P}} \text{ which fulfill Def. 18.1, holds } \dot{\mathcal{B}} \subseteq \dot{\mathcal{P}} \Leftrightarrow$$
$$\forall \mathcal{P} \text{ which fulfill Def. 17.1, holds } \mathcal{B} \subseteq \mathcal{P}.$$

□

Therefore, there exists exactly one base for each test knowledge space, if Q is finite.

The base is the most compressed form for storing the list of test knowledge states. By means of the base $\dot{\mathcal{B}}$ we can infer the test knowledge space $\dot{\mathcal{K}}$, the corresponding knowledge space $\mathcal{K}$ and the surmise relation between items. Moreover - and this is an important conclusion of our concept - we can also infer the surmise relation between tests and its properties as there are anti-symmetry, transitivity, left- and right-coveringness by means of the base.

Cor.20, Cor.21 and Cor.22 make it very easy to investigate the properties of the surmise relation between tests for quasi ordinal test knowledge spaces. Let in the following $\dot{\mathcal{B}} = \{\dot{K}_1, \ldots, \dot{K}_n\}$ for $i \in \{1, \ldots, n\}$ denote the base of the quasi ordinal test knowledge space $\dot{\mathcal{K}}$.

Corollary 20 $A \, \dot{\mathcal{S}} \, B \Leftrightarrow \forall \dot{K}_i \in \dot{\mathcal{B}}$ *with* $A_i = \emptyset : \bigcup B_i \subset B$.

PROOF.

$$\Rightarrow: A \, \dot{\mathcal{S}} \, B \Rightarrow \exists b \in B, \exists a \in A : a \in \bigcap \mathcal{K}_b \Rightarrow$$
$$\exists b \in B, a \in A : \forall K_i \in \mathcal{K} : (b \in K_i \Rightarrow a \in K_i) \Rightarrow$$
$$\exists b \in B, a \in A : \forall \dot{K}_i \in \dot{\mathcal{K}} : (b \in B_i = B \cap K_i \Rightarrow a \in A_i = A \cap K_i) \Rightarrow$$
$$\exists b \in B, a \in A : \forall \dot{K}_i \in \dot{\mathcal{K}} : (a \notin A_i \Rightarrow b \notin B_i) \Rightarrow$$

$\forall \dot{K}_i \in \dot{\mathcal{K}} : (A_i = \emptyset \Rightarrow B_i \neq B) \Rightarrow$
$\forall \dot{K}_i \in \dot{\mathcal{B}}$ with $A_i = \emptyset : \bigcup B_i \subset B$.
$\Leftarrow: \forall \dot{K}_i \in \dot{\mathcal{B}}$ with $A_i = \emptyset : \bigcup B_i \subset B \Rightarrow$
$\forall \dot{K}_i \in \dot{\mathcal{K}}$ with $A_i = \emptyset : B_i \neq B \Rightarrow$
$\forall \dot{K}_i \in \dot{\mathcal{K}}$ with $A_i = \emptyset : \exists b \in B$ with $b \notin B_i$.
Supposition 1: $\forall b \in B \;\; \exists \dot{K}_i \in \dot{\mathcal{K}}$ with $A_i = \emptyset \wedge b \in B_i \Rightarrow$
For $\dot{K}_k := \dot{\bigcup\limits_{A_i=\emptyset}} \dot{K}_i$ we have $A_k = \emptyset, B_k = B \wedge \dot{K}_k \in \dot{\mathcal{K}}$,
as $\dot{\mathcal{K}}$ is closed under union .
This is a contradiction to our assumption $\Rightarrow$
Supposition 1 is wrong $\Rightarrow$
$\exists b \in B : \forall \dot{K}_i \in \dot{\mathcal{K}}$ with $A_i = \emptyset : b \notin B_i \Rightarrow$
$\exists b \in B : \forall \dot{K}_i \in \dot{\mathcal{K}}$ with $b \in B_i : A_i \neq \emptyset \Rightarrow$
$\exists b \in B : \forall \dot{K}_i \in \dot{\mathcal{K}}$ with $b \in B_i \exists a \in A : a \in A_i$. $\quad (*)$
Supposition 2: $\forall a \in A \;\; \exists \dot{K}_i : b \in B_i \wedge a \notin A_i \Rightarrow$
For $\dot{K}_t := \dot{\bigcap\limits_{b \in B_i}} \dot{K}_i$ we have $A_t = \emptyset, b \in B_t \wedge K_t \in \dot{\mathcal{K}}$
as $\dot{\mathcal{K}}$ is closed under intersection.
This is a contradiction to (*) $\Rightarrow$
Supposition 2 is wrong $\Rightarrow$
$\exists b \in B, a \in A : \forall \dot{K}_i \in \dot{\mathcal{K}} : (b \in B_i \Rightarrow a \in A_i) \Rightarrow$
$A \; \dot{\mathcal{S}} \; B$.

□

Corollary 21 $A \; \dot{\mathcal{S}}_l \; B \Leftrightarrow \forall \dot{K}_i \in \dot{\mathcal{B}}$ *with* $B_i \neq \emptyset : A_i \neq \emptyset$

PROOF.

$\Rightarrow: A \; \dot{\mathcal{S}}_l \; B \Rightarrow \forall b \in B : A_b \neq \emptyset \Rightarrow$
$\forall b \in B \;\; \exists a \in A : a \in \bigcap \mathcal{K}_b \Rightarrow$
$\forall K_i \in \mathcal{K} : (\exists b \in B \cap K_i \Rightarrow \exists a \in A \cap K_i) \Rightarrow$
$\forall K_i \in \mathcal{K} : (B_i \neq \emptyset \Rightarrow A_i \neq \emptyset) \Rightarrow$
$\forall \dot{K}_i \in \dot{\mathcal{K}} : (B_i \neq \emptyset \Rightarrow A_i \neq \emptyset) \Rightarrow$
$\forall \dot{K}_i \in \dot{\mathcal{B}} : (B_i \neq \emptyset \Rightarrow A_i \neq \emptyset)$.
$\Leftarrow \forall \dot{K}_i \in \dot{\mathcal{B}} : (B_i \neq \emptyset \Rightarrow A_i \neq \emptyset)$.
Supposition: $\exists b \in B : A_b = \emptyset \Rightarrow$

$\exists b \in B : \forall a \in A : a \notin \bigcap \mathcal{K}_b \Rightarrow$
$\exists b \in B : \forall a \in A \;\; \exists K_i \in \mathcal{K} : b \in K_i \wedge a \notin K_i \Rightarrow$
$\exists b \in B : \forall a \in A \;\; \exists \dot{K}_i \in \dot{\mathcal{K}} : b \in B_i = B \cap K_i \wedge a \notin A_i = a \cap K_i \Rightarrow$
$\exists b \in B :$ for $\dot{K}_j := \dot{\bigcap\limits_{b \in B_i}}$ we have $A_i = \emptyset \wedge B_i \neq \emptyset($ as $b \in B_i)$.
This is a contradiction to our assumption $\Rightarrow$
The Supposition is wrong $\Rightarrow$
$\forall b \in B : A_b \neq \emptyset \Rightarrow A \, \dot{\mathcal{S}}_l \, B.$

□

Corollary 22 *$A \, \dot{\mathcal{S}}_r \, B \Leftrightarrow \forall \dot{K}_1, \ldots, \dot{K}_n \in \dot{\mathcal{B}}$ with $\dot{\bigcup\limits_{i=1}^{n}} B_i = B : \dot{\bigcup\limits_{i=1}^{n}} A_i = A$*

PROOF.

$\Rightarrow A \, \dot{\mathcal{S}}_r \, B \Rightarrow \bigcup\limits_{b \in B} A_b = A \Rightarrow$
$\bigcup\limits_{b \in B} \left(A \cap \bigcap \mathcal{K}_b \right) = A \Rightarrow$
$A \cap \left(\bigcup\limits_{b \in B} \left(\bigcap \mathcal{K}_b \right) \right) = A \Rightarrow$
$A \subseteq \bigcup\limits_{b \in B} \left(\bigcap \mathcal{K}_b \right) \Rightarrow$
$\forall a \in A \exists b \in B : a \in \bigcap \mathcal{K}_b.$
Suppose $\dot{K}_1, \ldots, \dot{K}_n \in \dot{\mathcal{B}}$ with $\dot{\bigcup\limits_{j=1}^{n}} B_j = B \Rightarrow$
For $\dot{K}_i := \dot{K}_1 \dot{\cup} \ldots \dot{\cup} K_j \in \dot{\mathcal{K}} \;\; \dot{\bigcup\limits_{j=1}^{n}} B_j = B \Rightarrow$
$\forall b \in B : b \in B_i \Rightarrow$
$\forall a \in A \;\; \exists b \in B : b \in K_i \wedge a \in \bigcap \mathcal{K}_b \Rightarrow$
$\forall a \in A a \in K_i \Rightarrow A_i = A.$
$\Leftarrow : \forall \dot{K}_1, \ldots, \dot{K}_n \in \dot{\mathcal{B}} : \left(\dot{\bigcup\limits_{j=1}^{n}} B_j = B \Rightarrow \dot{\bigcup\limits_{j=1}^{n}} A_j = A \right) \Rightarrow$
$\forall \dot{K}_i \in \dot{\mathcal{K}} : (B_i = B \Rightarrow A_i = A) \quad (*).$
Supposition: $\exists a \in A : \forall b \in B : a \notin \bigcap \mathcal{K}_b \Rightarrow$
$\exists a \in A \;\; \forall b \in B \;\; \exists K_i \in \mathcal{K} : b \in K_i \wedge a \notin K_i \Rightarrow$
$\exists a \in A \;\; \forall b \in B \;\; \exists \dot{K}_i \in \dot{\mathcal{K}} : b \in B_i \wedge a \notin A_i.$

For $\dot{K}_k := \dot{\bigcup\limits_{a \notin A_i}}$ we have $B_k = B, A_k \neq A($ as $a \notin A_k)$
$\wedge \dot{K}_k \in \dot{\mathcal{K}},$ as $\dot{\mathcal{K}}$ is closed under union.
This is a contradiction to (*) $\Rightarrow$
The supposition is wrong $\Rightarrow$
$\forall a \in A \;\; \exists b \in B : a \in \bigcap \mathcal{K}_b \Rightarrow$

$$\forall a \in A : a \in \left(\bigcup_{b \in B} \left(\bigcap \mathcal{K}_b \right) \right) \Rightarrow$$

$$A \cap \left(\bigcup_{b \in B} \left(\bigcap \mathcal{K}_b \right) \right) = A \Rightarrow$$

$$\bigcup_{b \in B} A_b = A \Rightarrow A \; \dot{\mathcal{S}}_r \; B.$$

□

5 Further research

Based upon these current results we now want to find efficient ways of partitioning sets of items into tests regarding mathematcal criteria as antisymmetry, transitivity and left- and right-coveringness as well as content-oriented criteria.

Furthermore, we want to investigate interdependencies and parallelity for tests. There are different levels of parallelity, in the following two extreme cases are shown:

- *weak parallelity:* $A \; \dot{\mathcal{S}} \; B \wedge B \; \dot{\mathcal{S}} \; A$ (See Figure 9).
- *strong parallelity:* $A \; \dot{\mathcal{S}}_l \; B \wedge B \; \dot{\mathcal{S}}_l \; A$ (See Figure 10).

The question occurs, which criteria for unifiability shall be choosen.

Furthermore we want to generalize the concept of surmise relations between tests to surmise *systems* between tests, which allow different ways of solving a problem.

Finally we want to establish principles for handling data - especially noisy data. In general, empirically obtained data are noisy, e.g. because of careless errors and lucky guesses or because of missing data. Methods for handling such data must be found.

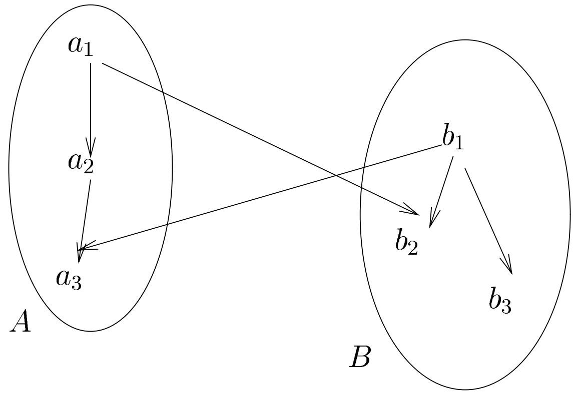

Fig. 9. $A \dot{\mathcal{S}} B \wedge B \dot{\mathcal{S}} A$

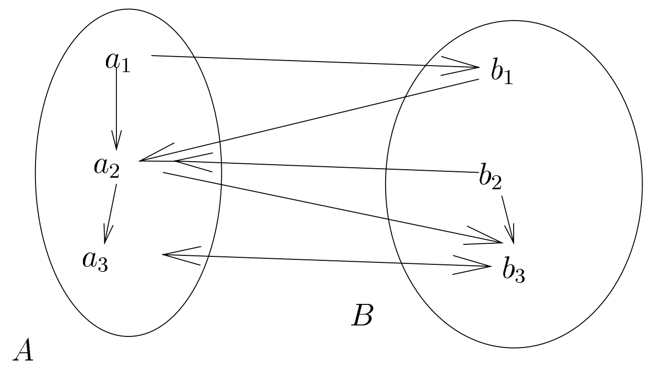

Fig. 10. $A \dot{\mathcal{S}}_l B \wedge B \dot{\mathcal{S}}_l A$

References

[1] Dietrich Albert, editor. *Knowledge Structures.* Springer Verlag, New York, 1994.

[2] Dietrich Albert. Surmise relations between tests. Talk at the 28th Annual Meeting of the Society for Mathematical Psychology, University of California, Irvine, August 1995, 1995.

[3] Jean-Paul Doignon and Jean-Claude Falmagne. Spaces for the assessment of knowledge. *International Journal of Man-Machine Studies*, 23:175–196, 1985.

[4] Jean-Paul Doignon and Jean-Claude Falmagne. *Knowledge Spaces.* Springer–Verlag, Berlin, 1998.

[5] Jean-Claude Falmagne, Mathieu Koppen, Mike Villano, Jean-Paul Doignon, and Leila Johannesen. Introduction to knowledge spaces: how to build, test and search them. *Psychological Review*, 97:201–224, 1990.