Ünlü, A., Brandt, S., & Albert, D. (2004). Test surmise relations, test knowledge structures, and their characterizations. *Discrete Applied Mathematics*, submitted for publication, 76 pages.

TEST SURMISE RELATIONS, TEST KNOWLEDGE STRUCTURES, AND THEIR CHARACTERIZATIONS

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Abstract

This paper investigates natural, left-, right-, and total-covering test surmise relations on a set of tests partitioning the domain of a knowledge structure. The properties of reflexivity, transitivity, and antisymmetry are examined. In particular, it is shown that the property of antisymmetry is satisfied for the left-, right-, and total-covering test surmise relations when the underlying knowledge structure is discriminative and the domain is finite. This paper also investigates natural, l-, r-, and c-type test knowledge structures. The concepts of a test surmise relation and test knowledge structure respectively generalize the concepts of a surmise relation and knowledge structure in knowledge space theory. The main thrust of this paper is an examination of characterizations of these models. Unlike at the level of items, at the level of tests, the test surmise relations and test knowledge structures may not necessarily be (uniquely) derived from each other. (a) Each can be characterized by the underlying surmise relation and knowledge structure, (b) the test surmise relations can be characterized by the test knowledge structures, and (c) the test knowledge structures can, at least under some condition, be characterized by the test surmise relations.

Key words: Knowledge space theory, Surmise relation, Knowledge structure, Test surmise relation, Test knowledge structure, Reflexivity, Transitivity, Antisymmetry, Characterization

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Contents

1 Introduction

1.1 Motivation

[1] This work belongs to the field of knowledge space theory (KST) which was introduced by Doignon and Falmagne (1985, 1999). We recapitulate some of the relevant basic concepts of KST in Section 2. In this theory, a body of information (e.g., high school mathematics) is represented by a set Q of (dichotomous) items giving a representative coverage of the information body. On the set Q, two types of models are considered, knowledge structures and surmise relations. Knowledge structures and surmise relations are defined as families of subsets of Q containing at least the empty set $\emptyset$ and Q, respectively, as reflexive and transitive binary relations on Q. These models have been successfully utilized, for instance, for the adaptive assessment and training of knowledge (e.g., Doignon & Falmagne, 1999).

Often it is desirable to partition a body of information into special sub-bodies:

1. In curriculum development (e.g., Albert & Hockemeyer, 1999), for instance in high school mathematics, the entire body of information (represented by Q) can be partitioned into such sub-bodies (represented by subsets of Q) as algebra, analysis, and geometry. An analysis of a student's knowledge at the level of subsets of items (tests) is useful because there are natural breaks in an academic subject, in particular in high school mathematics, around which curricula can be arranged. Work is also being done in the development

[1] Further motivation for the research reported in this paper can be found in Subsections 7.2 and 7.3.

of efficient tutoring systems that improve curriculum efficiency (e.g., Albert & Hockemeyer, 1999), to which the present study might be applied as well.

2. In the case of the well-established computer educational tutorial system `ALEKS`[2], the study of tests is motivated by the following factors: (a) instructors moving through standard, entrenched curricula are looking to have their students work on 'units' or 'modules' of course material, and the software has to be able to accommodate this; (b) any 'global' assessment on the entire domain must take into account the performance on these modules; and (c) the underlying surmise relation (at the level of individual items) may be too small in cardinality—there may be too few inferences possible—to allow for a global assessment of appropriate length, so assessments must be 'pieced together.'

All of this is to say that the problem of dividing a curriculum, that is, the study of tests, even at the level of a high school course such as geometry (and not the entire high school curriculum), is a real one, and in particular, it is one that requires an eventual consideration of probabilistic, as well as combinatorial, properties.

1.2 Brief conceptual overview of the paper

This paper considers combinatorial, not probabilistic, properties; see Section 7 for a discussion of further extensions and modifications of the present approach, and a concluding resume. We investigate some possible relationships among 'tests' of items in a knowledge structure, with a test being an element of a partition of the domain of the knowledge structure.

[2] Assessment and LEarning in Knowledge Spaces: `http://www.aleks.com/`.

Surmise relations among tests, based on the underlying surmise relation among items, are examined. In particular, relations defined as follows are discussed: For two tests A and B,

(B, A) *is in the relation* if and only if (iff) there is some $b \in B$ that is a prerequisite for some $a \in A$;

(B, A) *is in the relation* iff each $a \in A$ has some prerequisite $b \in B$;

(B, A) *is in the relation* iff each $b \in B$ is a prerequisite for some $a \in A$;

(B, A) *is in the relation* iff each $a \in A$ has some prerequisite $b \in B$ and each $b \in B$ is a prerequisite for some $a \in A$.

Given a partition made up of tests for a knowledge structure, one can obtain a 'test knowledge structure' via intersections of the tests with the original states in the underlying knowledge structure. We investigate different types of test knowledge structures, which differ according to whether the student is required to know some or all of the items in the intersection, to be assigned a certain state.

The main thrust of this paper is an examination of characterizations of the test surmise relations and test knowledge structures. Unlike at the level of items (cf. Birkhoff's, 1937, Theorem 8), at the level of tests, the test surmise relations and test knowledge structures may not necessarily be (uniquely) derived from each other, let alone be set in a one-to-one correspondence. However, (a) each can be characterized by the underlying surmise relation and knowledge structure at the level of items, (b) the test surmise relations can (not necessarily uniquely) be characterized by the test knowledge structures, and (c) the test knowledge structures can at least 'partly' (under some condition) be characterized by the test surmise relations.

This paper extends the work by Brandt, Albert, and Hockemeyer (2003). The examined test surmise relations and test knowledge structures respectively generalize the common surmise relation and knowledge structure models at the level of items in KST.

1.3 Structure of the paper

Basic concepts of KST are reviewed (Section 2). The natural, left-, right-, and total-covering test surmise relations are discussed (Section 3). The properties of reflexivity, transitivity, and antisymmetry are investigated; in particular, it is shown that the property of antisymmetry is satisfied for the left-, right-, and total-covering test surmise relations in the case of an underlying discriminative knowledge structure on a finite domain. The natural, l-, r-, and c-type test knowledge structures are discussed (Section 4). Characterizations of the test surmise relations are obtained (Section 5). The test surmise relations can (not necessarily uniquely) be characterized by the test knowledge structures. Characterizations of the test knowledge structures are obtained (Section 6). Contrary to the test surmise relations, the test knowledge structures may not necessarily be characterized by the test surmise relations. However, a sufficient condition is proposed under which some characterizations of the test knowledge structures via the test surmise relations do hold. This paper ends with a discussion containing a summary, some suggestions for further extensions and modifications, and a concluding resume (Section 7).

2 Basic concepts of knowledge space theory

In 1985, Jean-Paul Doignon and Jean-Claude Falmagne introduced *knowledge space theory* (KST; Doignon & Falmagne, 1985). Most of KST is presented in the monograph 'Knowledge Spaces' by Doignon and Falmagne (1999); see also Doignon and Falmagne (1987), Falmagne (1989), and Falmagne, Koppen, Villano, Doignon, and Johannesen (1990). For application examples, see in particular Albert and Lukas (1999). This section briefly reviews some of the relevant basic concepts of KST; for details, refer to the afore mentioned references.

2.1 Knowledge structures and spaces

A general concept is that of a *knowledge structure.*

Definition 1 *A knowledge structure is a pair $(Q, \mathcal{K})$, with Q a non-empty, finite set, and $\mathcal{K}$ a family of subsets of Q containing at least $\emptyset$ and Q. The set Q is called the domain of the knowledge structure. The elements $q \in Q$ and $K \in \mathcal{K}$ are referred to as items and knowledge states, respectively. One also says that $\mathcal{K}$ is a knowledge structure on Q.*

The general definition of a knowledge structure allows for infinite item sets as well. Throughout this paper, however, Q is assumed to be finite.

Knowledge spaces are special knowledge structures.

Definition 2 *A knowledge structure $(Q, \mathcal{K})$ is called a knowledge space iff $\mathcal{K}$ is closed under union, that is, for all $\mathcal{F} \subset \mathcal{K}$, $\bigcup \mathcal{F} \in \mathcal{K}$. A knowledge space*

$(Q, \mathcal{K})$ is called quasi ordinal iff $\mathcal{K}$ is closed under intersection, that is, for all $\mathcal{F} \subset \mathcal{K}$, $\bigcap \mathcal{F} \in \mathcal{K}$.

2.2 Discriminativity

Next the concept of a *notion* is defined.

Definition 3 *Let $(Q, \mathcal{K})$ be a knowledge structure. For any $q \in Q$, let $\mathcal{K}_q := \{K \in \mathcal{K} : q \in K\}$ denote the set of all knowledge states containing q. The set $q^* := \{r \in Q : \mathcal{K}_r = \mathcal{K}_q\}$, which consists of all items that are contained in exactly the same knowledge states as item q, is called a notion.*

In other words, an item $r \in Q$ belongs to the notion q^* $(q \in Q)$ iff every knowledge state which contains q also contains r, and vice versa. Stated differently, whenever a person (latently) masters item q, she/he will also be able to master item r, and vice versa. In testing the knowledge of an examinee, only one of these two questions needs to be asked. Two items belonging to the same notion are called *equally informative.*

This leads to the concept of *discriminativity.*

Definition 4 *A knowledge structure with each of its notions a singleton is called discriminative. A discriminative quasi ordinal knowledge space is called an ordinal knowledge space.*

The next lemma states that the notions of a knowledge structure form a *partition* of the domain, and that any knowledge structure determines a corresponding discriminative knowledge structure.

Lemma 5 *Let $(Q, \mathcal{K})$ be a knowledge structure.*

1. *The set Q^* of all notions is a partition of Q, that is, Q^* is a family of pairwise disjoint (and non-empty) subsets of Q with $\bigcup Q^* = Q$.*
2. *A discriminative knowledge structure can be derived from $(Q, \mathcal{K})$ as follows. Form the set Q^* of all notions in $(Q, \mathcal{K})$ and define $\mathcal{K}^* := \{K^* : K \in \mathcal{K}\}$ where $K^* := \{q^* : q \in K\}$ $(K \in \mathcal{K})$. Then $(Q^*, \mathcal{K}^*)$ is a discriminative knowledge structure, called the discriminative reduction of $(Q, \mathcal{K})$.* □

2.3 Surmise relations and Birkhoff's theorem

Any knowledge structure determines a quasi order, its *surmise relation.*

Definition 6 *Let $(Q, \mathcal{K})$ be a knowledge structure. The relation $S \subset Q \times Q$ defined by $rSq :\Longleftrightarrow r \in \bigcap \mathcal{K}_q$ $(q, r \in Q)$ is called the surmise relation of the knowledge structure.*

In other words, rSq iff r is an element of all knowledge states containing q. Stated differently, each person mastering problem q is also able to master problem r (Figure 1). Item r is a prerequisite for item q, in the sense that the mastery of r is a necessary condition for the mastery of q. Someone who is not able to master item r is also not able to master item q.

[Figure 1]

The next lemma summarizes properties of the surmise relation associated with a knowledge structure.

Lemma 7 *Let $(Q, \mathcal{K})$ be a knowledge structure. The surmise relation S as-*

sociated with the knowledge structure is a quasi order, that is, a reflexive, transitive binary relation on Q. If $(Q, \mathcal{K})$ is discriminative, S is even a partial order, that is, S also fulfills the property of antisymmetry.

Proof. For any two items $q, r \in Q$, it holds $rSq \Longleftrightarrow \mathcal{K}_r \supset \mathcal{K}_q$. □

Birkhoff's (1937) theorem provides a linkage between quasi ordinal knowledge spaces and surmise relations on an item set.

Theorem 8 (Birkhoff–Theorem) *There exists a one-to-one correspondence between the family of all quasi ordinal knowledge spaces $\mathcal{K}$ on a domain Q, and the family of all surmise relations S on Q. Such a correspondence is defined through the two equivalences ($p, q \in Q$, $K \subset Q$):*

$$\begin{aligned} pSq \; &:\Longleftrightarrow \; \left[\forall K \in \mathcal{K} : \left\{ q \in K \Longrightarrow p \in K \right\}\right], \\ K \in \mathcal{K} \; &:\Longleftrightarrow \; \left[\forall (pSq) : \left\{ q \in K \Longrightarrow p \in K \right\}\right]. \end{aligned}$$

Under this correspondence ordinal knowledge spaces bijectively correspond to partial orders.

Proof. See Doignon and Falmagne (1999, pp. 39–40, Theorem 1.49). □

3 Test surmise relations

In KST single items and surmise relations among them are considered. This is generalized in this section which discusses test surmise relations, binary relations among subsets, tests, of items. In fact, 'test surmise relation' is a generic term which stands for four types of binary relations among tests, the natural, left-, right-, and total-covering test surmise relations.

3.1 Tests

As mentioned in Subsection 1.1, it is often desirable to partition a body of information (represented by a domain Q) into special sub-bodies. This can be formalized by a *partition* $\mathcal{T}$ of the domain Q, that is, $\mathcal{T}$ is a family of subsets of Q such that (a) $Q = \cup\mathcal{T}$, (b) $T \neq \emptyset$ for all $T \in \mathcal{T}$, and (c) $T \cap T' = \emptyset$ for all $T, T' \in \mathcal{T}$, $T \neq T'$.

Definition 9 *Let $(Q, \mathcal{K})$ be a knowledge structure and $\mathcal{T}$ a partition of Q. The elements of $\mathcal{T}$ are called tests.*

A notation that is used when defining test surmise relations:

Definition 10 *Let $(Q, \mathcal{K})$ be a knowledge structure, and let $\mathcal{T}$ be a set of tests in Q. For any $q \in Q$ and $B \in \mathcal{T}$, let $B_q := B \cap \bigcap \mathcal{K}_q$ ($\mathcal{K}_q := \{K \in \mathcal{K} : q \in K\}$).*

3.2 Natural test surmise relations

The first type of binary relation among tests is the *natural test surmise relation.*

Definition 11 *Let $(Q, \mathcal{K})$ be a knowledge structure, and let $\mathcal{T}$ be a set of tests in Q. The binary relation $\dot{\mathcal{S}} \subset \mathcal{T} \times \mathcal{T}$ on $\mathcal{T}$ defined by ($A, B \in \mathcal{T}$)*

$$B \, \dot{\mathcal{S}} \, A :\Longleftrightarrow \Big[\exists a \in A : B_a \neq \emptyset \Big]$$

is called the natural test surmise relation (NTSR) associated with $(Q, \mathcal{K})$ and $\mathcal{T}$.

NTSRs are interpreted as follows. From a person's mastery of an appropriate item in test A, one can surmise the mastery of a non-empty subset of items in test B (Figure 2). Mastering these 'test-B' items is a prerequisite for mastering this 'test-A' item. In other words, from a particular mastery on test A a minimum mastery on test B can be surmised.

[Figure 2]

The next corollary is a consequence of Definitions 6 and 11, and it states that the NTSR among tests can be characterized by (derived from) the underlying surmise relation among items.

Corollary 12 *Let $(Q, \mathcal{K})$ be a knowledge structure and $\mathcal{T}$ a set of tests in Q. Let S be the surmise relation (among items) associated with $(Q, \mathcal{K})$. Let $\dot{\mathcal{S}}$ be the NTSR (among tests) associated with $(Q, \mathcal{K})$ and $\mathcal{T}$. Then, for any $A, B \in \mathcal{T}$, it holds $B \,\dot{\mathcal{S}}\, A \Longleftrightarrow \left[\exists a \in A \, \exists b \in B : bSa\right]$.* □

Next the properties of NTSRs are summarized.

Proposition 13 *Natural test surmise relations are reflexive, but they are neither transitive nor antisymmetric in general.*

Proof. Reflexivity: Let $A \in \mathcal{T}$. Since $A \neq \emptyset$, there is an element $a \in A$. Then $a \in A \cap \bigcap \mathcal{K}_a =: A_a$. Thus $A_a \neq \emptyset$. This proves $A \,\dot{\mathcal{S}}\, A$ (Figure 3).

[Figure 3]

Transitivity: Consider on $Q := \{a, b, c, d\}$ the quasi order $S \subset Q \times Q$,[3]

$$S := \triangle_{Q \times Q} \cup \{(c, a), (d, b)\}.$$

[3] Let $\triangle_{Q \times Q} := \{(x, y) \in Q \times Q : x = y\}$ denote the diagonal in $Q \times Q$.

(According to Theorem 8, this quasi order uniquely corresponds to a knowledge structure, in the sense of Definition 6.) Consider the partition $\mathcal{T}$ consisting of the tests $A := \{a\}$, $B := \{b, c\}$, and $C := \{d\}$. For the NTSR $\dot{\mathcal{S}} \subset \mathcal{T} \times \mathcal{T}$, it holds $C \,\dot{\mathcal{S}}\, B$ and $B \,\dot{\mathcal{S}}\, A$, but $C \,\not\dot{\mathcal{S}}\, A$ (Figure 4).

[Figure 4]

Antisymmetry: Consider on $Q := \{a, b, c, d\}$ the quasi order $S \subset Q \times Q$,

$$S := \triangle_{Q\times Q} \cup \{(d, a), (b, c)\}.$$

Consider the family $\mathcal{T}$ of tests $A := \{a, b\}$ and $B := \{c, d\}$. For the NTSR $\dot{\mathcal{S}}$, it holds $B \,\dot{\mathcal{S}}\, A$ and $A \,\dot{\mathcal{S}}\, B$, but $A \neq B$ (Figure 5).

[Figure 5]

□

Hence natural test surmise relations are not necessarily quasi orders. In particular, they are not necessarily partial orders. In the following subsections, however, we describe types of binary relations among tests which are quasi orders, the *left-*, *right-*, and *total-covering test surmise relations.*

3.3 Left-covering test surmise relations

Definition 14 *Let $(Q, \mathcal{K})$ be a knowledge structure, and let $\mathcal{T}$ be a set of tests in Q. The binary relation $\dot{\mathcal{S}}_l \subset \mathcal{T} \times \mathcal{T}$ on $\mathcal{T}$ defined by $(A, B \in \mathcal{T})$*

$$B \,\dot{\mathcal{S}}_l\, A :\Longleftrightarrow \Big[\forall a \in A : B_a \neq \emptyset\Big]$$

is called the left-covering test surmise relation (LTSR) associated with $(Q, \mathcal{K})$ *and* $\mathcal{T}$.

LTSRs are interpreted as follows. From the mastery of any item in test A, one can surmise the mastery of a non-empty subset of items in test B (Figure 6). Mastering these 'test-B' items altogether is a prerequisite for mastering the entire test A. In other words, from any mastery on test A a minimum mastery on test B can be surmised.

[Figure 6]

The next corollary is a consequence of Definitions 6 and 14, and it states that the LTSR among tests can be characterized by (derived from) the underlying surmise relation among items.

Corollary 15 *Let* $(Q, \mathcal{K})$ *be a knowledge structure and* $\mathcal{T}$ *a set of tests in* Q. *Let* S *be the surmise relation associated with* $(Q, \mathcal{K})$. *Let* $\dot{\mathcal{S}}_l$ *be the LTSR associated with* $(Q, \mathcal{K})$ *and* $\mathcal{T}$. *Then, for any* $A, B \in \mathcal{T}$, *it holds* $B \, \dot{\mathcal{S}}_l \, A \Longleftrightarrow \Big[\forall a \in A \, \exists b \in B : bSa\Big]$. □

Next the properties of LTSRs are summarized.

Proposition 16 *Let* $(Q, \mathcal{K})$ *be a knowledge structure, and let* $\mathcal{T}$ *be a set of tests in* Q. *Let* $\dot{\mathcal{S}}$ *and* $\dot{\mathcal{S}}_l$ *denote the NTSR and LTSR associated with* $(Q, \mathcal{K})$ *and* $\mathcal{T}$, *respectively. It holds:*

1. $\dot{\mathcal{S}}_l \subset \dot{\mathcal{S}}$;
2. $\dot{\mathcal{S}}_l$ *is reflexive and transitive;*
3. $\dot{\mathcal{S}}_l$ *is not necessarily antisymmetric.*

Proof. 1. For any $B \; \dot{\mathcal{S}}_l \; A$ $(A, B \in \mathcal{T})$ and $a \in A$, it holds $B_a \neq \emptyset$.

2. Reflexivity: Let $A \in \mathcal{T}$ and $a \in A$. Then $a \in A \cap \bigcap \mathcal{K}_a =: A_a$, and $A_a \neq \emptyset$. Transitivity: Let $C \; \dot{\mathcal{S}}_l \; B$ and $B \; \dot{\mathcal{S}}_l \; A$ $(A, B, C \in \mathcal{T})$. Let $a \in A$. Since $B \; \dot{\mathcal{S}}_l \; A$, it holds $\emptyset \neq B_a := B \cap \bigcap \mathcal{K}_a$. Let $b \in B_a$. Since $C \; \dot{\mathcal{S}}_l \; B$, it holds $\emptyset \neq C_b := C \cap \bigcap \mathcal{K}_b$. This implies:

$$\emptyset \neq C \cap \bigcap \mathcal{K}_b \subset C \cap \bigcap \mathcal{K}_a =: C_a.$$

This is illustrated in Figure 7.

[Figure 7]

3. Let $Q := \{a, b\}$, $\mathcal{K} := \{\emptyset, Q\}$, and $\mathcal{T} := \Big\{A := \{a\}, B := \{b\}\Big\}$. Then $A \; \dot{\mathcal{S}}_l \; B$ and $B \; \dot{\mathcal{S}}_l \; A$, but $A \neq B$. □

3.4 Right-covering test surmise relations

Right-covering test surmise relations are another type of binary relations among tests which are quasi orders.

Definition 17 *Let $(Q, \mathcal{K})$ be a knowledge structure, and let $\mathcal{T}$ be a set of tests in Q. The binary relation $\dot{\mathcal{S}}_r \subset \mathcal{T} \times \mathcal{T}$ on $\mathcal{T}$ defined by $(A, B \in \mathcal{T})$*

$$B \; \dot{\mathcal{S}}_r \; A :\Longleftrightarrow \left[\bigcup_{a \in A} B_a = B \right]$$

is called the right-covering test surmise relation (RTSR) associated with $(Q, \mathcal{K})$ and $\mathcal{T}$.

RTSRs are interpreted as follows. From the mastery of the entire test A the mastery of the entire test B can be surmised. The mastery of the entire test

B is a prerequisite for the mastery of the entire test A (Figure 8).

[Figure 8]

The next corollary is a consequence of Definitions 6 and 17, and it states that the RTSR among tests can be characterized by (derived from) the underlying surmise relation among items.

Corollary 18 *Let $(Q, \mathcal{K})$ be a knowledge structure and $\mathcal{T}$ a set of tests in Q. Let S be the surmise relation associated with $(Q, \mathcal{K})$. Let $\dot{\mathcal{S}}_r$ be the RTSR associated with $(Q, \mathcal{K})$ and $\mathcal{T}$. Then, for any $A, B \in \mathcal{T}$, it holds $B\ \dot{\mathcal{S}}_r\ A \Longleftrightarrow \Big[\forall b \in B\, \exists a \in A : bSa\Big]$.* □

Next the properties of RTSRs are summarized.

Proposition 19 *Let $(Q, \mathcal{K})$ be a knowledge structure, and let $\mathcal{T}$ be a set of tests in Q. Let $\dot{\mathcal{S}}$ and $\dot{\mathcal{S}}_r$ denote the NTSR and RTSR associated with $(Q, \mathcal{K})$ and $\mathcal{T}$, respectively. It holds:*

1. *$\dot{\mathcal{S}}_r \subset \dot{\mathcal{S}}$;*
2. *$\dot{\mathcal{S}}_r$ is reflexive and transitive;*
3. *$\dot{\mathcal{S}}_r$ is not necessarily antisymmetric.*

Proof. 1. Let $B\ \dot{\mathcal{S}}_r\ A$ $(A, B \in \mathcal{T})$. Since $\bigcup_{a \in A} B_a = B$ and $B \neq \emptyset$, there exists an element $a \in A$ with $B_a \neq \emptyset$.
2. Reflexivity: For any $A \in \mathcal{T}$, it holds $\bigcup_{a \in A} A_a = A$.
Transitivity: Let $C\ \dot{\mathcal{S}}_r\ B$ and $B\ \dot{\mathcal{S}}_r\ A$ $(A, B, C \in \mathcal{T})$. Let $c \in C$. Since $\bigcup_{b \in B} C_b = C$, there is an element $b \in B$ with $c \in C_b$. Since $\bigcup_{a \in A} B_a = B$, there is an element $a \in A$ with $b \in B_a$. Hence $\mathcal{K}_a \subset \mathcal{K}_b \subset \mathcal{K}_c$, and it follows

that $c \in C \cap \bigcap \mathcal{K}_c \subset C \cap \bigcap \mathcal{K}_a =: C_a$ (Figure 9).

[Figure 9]

3. Let $Q := \{a, b\}$, $\mathcal{K} := \{\emptyset, Q\}$, and $\mathcal{T} := \Big\{A := \{a\}, B := \{b\}\Big\}$. Then $A \, \dot{\mathcal{S}}_r \, B$ and $B \, \dot{\mathcal{S}}_r \, A$, but $A \neq B$. □

3.5 Total-covering test surmise relations

The intersection of the LTSR and the RTSR is the *total-covering test surmise relation.*

Definition 20 *Let $(Q, \mathcal{K})$ be a knowledge structure, and let $\mathcal{T}$ be a set of tests in Q. Let $\dot{\mathcal{S}}_l$ and $\dot{\mathcal{S}}_r$ be the LTSR and RTSR associated with $(Q, \mathcal{K})$ and $\mathcal{T}$, respectively. The binary relation $\dot{\mathcal{S}}_t := \dot{\mathcal{S}}_l \bigcap \dot{\mathcal{S}}_r$ on $\mathcal{T}$ is called the total-covering test surmise relation (TTSR) associated with $(Q, \mathcal{K})$ and $\mathcal{T}$.*

TTSRs combine both the interpretations of LTSRs and RTSRs (Figure 10).

[Figure 10]

From Corollaries 15 and 18, it follows that the TTSR $\dot{\mathcal{S}}_t$ can be characterized by (derived from) the underlying surmise relation S $(A, B \in \mathcal{T})$:

$$B \, \dot{\mathcal{S}}_t \, A \Longleftrightarrow \Big[\Big[\forall a \in A \, \exists b \in B : bSa\Big] \text{ and } \Big[\forall b \in B \, \exists a \in A : bSa\Big]\Big].$$

Moreover, Propositions 16 and 19 imply that (a) the TTSR is a subset of the LTSR, RTSR, and NTSR, (b) the TTSR is reflexive and transitive, and (c) the TTSR is not necessarily antisymmetric.

Lemma 7 states that the surmise relation associated with a *discriminative* knowledge structure is a partial order, that is, it satisfies the property of antisymmetry. In this subsection, we investigate this property for the introduced test surmise relations. It is shown that discriminativity of the underlying knowledge structure (item level) can imply the property of antisymmetry for the associated left-, right-, and total-covering test surmise relations. This is definitely true for the case of a finite domain. If the domain is not finite, discriminativity of the underlying knowledge structure may not imply the property of antisymmetry for the respective test surmise relations. Finally, natural test surmise relations are not antisymmetric in general, even in the case of a discriminative knowledge structure on a finite domain.

Proposition 21 *Let $(Q, \mathcal{K})$ be a discriminative knowledge structure in which Q is a non-empty and finite set, and let $\mathcal{T}$ be a set of tests in Q. The associated left-, right-, and total-covering test surmise relations $\dot{\mathcal{S}}_l$, $\dot{\mathcal{S}}_r$, and $\dot{\mathcal{S}}_t$, respectively, are partial orders on $\mathcal{T}$.*

Proof. Let S be the surmise relation of $(Q, \mathcal{K})$, which is a partial order on Q. Let $A \; \dot{\mathcal{S}}_l \; B$ and $B \; \dot{\mathcal{S}}_l \; A$ $(A, B \in \mathcal{T})$. Assume that $A \neq B$. Since A and B are tests, it holds $A \neq \emptyset$, $B \neq \emptyset$, and $A \cap B = \emptyset$. Let $a \in A$. Because $A \; \dot{\mathcal{S}}_l \; B$ and $B \; \dot{\mathcal{S}}_l \; A$, there is a sequence $\{(a_n, b_n)\}_{n \in \mathbb{N}}$ in $A \times B$ such that

$a_1 := a$,
$b_1 \in B_{a_1}$,
for any $n \geq 2$, $a_n \in A_{b_{n-1}}$ and $b_n \in B_{a_n}$.

Then $a_{n+1}Sb_nSa_n$ for all $n \in \mathbb{N}$ (i.e., $\cdots Sa_3Sb_2Sa_2Sb_1Sa_1$). Since we assume that $A \neq B$, it holds $a_n \neq a_{n'}$ for all $n, n' \in \mathbb{N}$, $n \neq n'$. (If there exists $n, n' \in \mathbb{N}$, $n < n'$ with $a_n = a_{n'}$, then $a_{n'}Sb_{n'-1}Sa_n$. Because S is antisymmetric, it follows $b_{n'-1} = a_n$, and hence $A \cap B \neq \emptyset$.) This implies that Q is infinite, contradicting $|Q| < \infty$.[4] The arguments in case of $\dot{\mathcal{S}}_r$ are similar. Finally, the property of antisymmetry obviously holds for $\dot{\mathcal{S}}_t$ $(:= \dot{\mathcal{S}}_l \cap \dot{\mathcal{S}}_r)$. □

Finiteness of the underlying domain is essential.

Lemma 22 *In Proposition 21, the condition $|Q| < \infty$ is essential. If Q is an infinite set, the statements of Proposition 21 do not hold in general.*

Proof. Let $Q := \mathbb{R}$ be the set of real numbers. Let $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ denote the sets of rational and irrational numbers, respectively. Consider the tests $A := \mathbb{Q}$ and $B := \mathbb{R} \setminus \mathbb{Q}$ in Q (i.e., $\mathcal{T} := \{A, B\}$). Let $\leq$ be the natural order relation on Q, which is a partial order. According to the Birkhoff–Theorem (Theorem 8), $\leq$ is the surmise relation associated with a discriminative knowledge structure $\mathcal{K}$ on Q. The NTSR, LTSR, RTSR, and TTSR associated with $(Q, \mathcal{K})$ and $\mathcal{T}$, however, turn out to be the same, namely, the Cartesian product $\mathcal{T} \times \mathcal{T}$. □

NTSRs are not antisymmetric in general, even in the case of a discriminative knowledge structure on a finite domain.

Lemma 23 *Let $(Q, \mathcal{K})$ be a discriminative knowledge structure in which Q is a non-empty and finite set, and let $\mathcal{T}$ be a set of tests in Q. The associated natural test surmise relation $\dot{\mathcal{S}}$ is not necessarily antisymmetric.*

[4] For a set X, let $|X|$ stand for the cardinality of X.

Proof. Consider on $Q := \{a, b, c\}$ the partial order $S \subset Q \times Q$,

$$S := \triangle_{Q \times Q} \cup \{(a, b), (a, c), (b, c)\}.$$

According to the Birkhoff–Theorem (Theorem 8), S is the surmise relation associated with a discriminative knowledge structure $\mathcal{K}$ on Q. Let the tests be $A := \{a, c\}$ and $B := \{b\}$ (i.e., $\mathcal{T} := \{A, B\}$). The NTSR $\dot{\mathcal{S}}$ associated with $(Q, \mathcal{K})$ and $\mathcal{T}$ is the Cartesian product $\mathcal{T} \times \mathcal{T}$. □

4 Test knowledge structures

In the previous section we have extended the concept of a surmise relation among individual items to that of a test surmise relation among special subsets of items, the tests. There is a natural way of transferring the concept of a knowledge structure at the level of items to that of a test knowledge structure. (In the following note that 'test knowledge structure' is a generic term which stands for the natural, l-type, r-type, and c-type test knowledge structures.)

4.1 Natural test knowledge structures

The *natural test knowledge state* of a person is defined as the $|\mathcal{T}|$-tuple of the subsets of the tests the person is capable of mastering.

Definition 24 *Let* $(Q, \mathcal{K})$ *be a knowledge structure, and let* $\mathcal{T}$ *be a set of tests in* Q*. The set*

$$\dot{\mathcal{K}} := \{\dot{K} := (T \cap K)_{T \in \mathcal{T}} : K \in \mathcal{K}\}$$

of all $|\mathcal{T}|$-tuples $\dot{K} := (T \cap K)_{T\in\mathcal{T}}$ ($K \in \mathcal{K}$) is called the natural test knowledge structure associated with $(Q,\mathcal{K})$ and $\mathcal{T}$. The $|\mathcal{T}|$-tuple $\dot{K} := (T \cap K)_{T\in\mathcal{T}}$ is called the natural test knowledge state corresponding to $K \in \mathcal{K}$.

If $K \in \mathcal{K}$ is the knowledge state of a person, $T \cap K$ for $T \in \mathcal{T}$ denotes the subset of items in test T the person is capable of mastering. In other words, the natural test knowledge state of a person contains information about which items in each of the tests are mastered by the person.

The next corollary is a consequence of Definition 24 and Theorem 8, and it states that the natural test knowledge structure can be characterized by (derived from) the underlying surmise relation among items.

Corollary 25 *Let $(Q,\mathcal{K})$ be a quasi ordinal knowledge space and $\mathcal{T}$ a set of tests in Q. Let S be the surmise relation associated with $(Q,\mathcal{K})$. Let $\dot{\mathcal{K}}$ be the natural test knowledge structure associated with $(Q,\mathcal{K})$ and $\mathcal{T}$. Then, for any $|\mathcal{T}|$-tuple $\mathbf{F} := (F_T)_{T\in\mathcal{T}}$ of subsets $F_T \subset T$ ($T \in \mathcal{T}$), it holds*

$$\mathbf{F} \in \dot{\mathcal{K}} \Longleftrightarrow \exists G \subset Q : \left\{ \Big[\forall (p,q) \in S : [q \in G \Longrightarrow p \in G]\Big] \right.$$
$$\left. \text{and } \Big[\forall T \in \mathcal{T} : T \cap G = F_T\Big] \right\}.$$

□

Next we introduce the concept of a (quasi ordinal) natural test knowledge space. For doing so, first we have to define the *union* and *intersection* of such $|\mathcal{T}|$-tuples, coordinate-wise, as the union and intersection of sets.

Definition 26 *Let $(Q,\mathcal{K})$ be a knowledge structure and $\mathcal{T}$ a set of tests in Q. Let $\dot{\mathcal{K}}$ be the natural test knowledge structure associated with $(Q,\mathcal{K})$ and $\mathcal{T}$.*

For any $\dot{K} := (T \cap K)_{T\in\mathcal{T}}$ *and* $\dot{K'} := (T \cap K')_{T\in\mathcal{T}}$ *(*$K, K' \in \mathcal{K}$*), we define*[5]

$$\dot{K}\dot{\cup}\dot{K'} := \Big(T \bigcap (K \cup K')\Big)_{T\in\mathcal{T}},$$
$$\dot{K}\dot{\cap}\dot{K'} := \Big(T \bigcap (K \cap K')\Big)_{T\in\mathcal{T}}.$$

More generally, let $\mathcal{F} := \{\dot{K} := (T \cap K)_{T\in\mathcal{T}} : K \in \mathcal{I}\}$ *(*$\mathcal{I} \subset \mathcal{K}$*) be a family in* $\dot{\mathcal{K}}$*. The union and intersection of* $\mathcal{F}$ *are respectively defined by*

$$\dot{\bigcup}\mathcal{F} := \Big(T \bigcap \cup_{K\in\mathcal{I}} K\Big)_{T\in\mathcal{T}},$$
$$\dot{\bigcap}\mathcal{F} := \Big(T \bigcap \cap_{K\in\mathcal{I}} K\Big)_{T\in\mathcal{T}}.$$

Natural test knowledge spaces are special natural test knowledge structures.

Definition 27 *A natural test knowledge structure* $\dot{\mathcal{K}}$ *is called a natural test knowledge space iff* $\dot{\mathcal{K}}$ *is closed under* $\dot{\cup}$*-union, that is, for all* $\mathcal{F} \subset \dot{\mathcal{K}}$*,* $\dot{\bigcup}\mathcal{F} \in \dot{\mathcal{K}}$*. A natural test knowledge space* $\dot{\mathcal{K}}$ *is called quasi ordinal iff* $\dot{\mathcal{K}}$ *is closed under* $\dot{\cap}$*-intersection, that is, for all* $\mathcal{F} \subset \dot{\mathcal{K}}$*,* $\dot{\bigcap}\mathcal{F} \in \dot{\mathcal{K}}$*.*

The next corollary is a consequence of the previous definitions.

Corollary 28 *Let* $\dot{\mathcal{K}}$ *be a natural test knowledge structure associated with a knowledge structure* $(Q, \mathcal{K})$ *and a set* $\mathcal{T}$ *of tests in* Q*. Then,* $\dot{\mathcal{K}}$ *is a natural test knowledge space iff* $\mathcal{K}$ *is a knowledge space;* $\dot{\mathcal{K}}$ *is a quasi ordinal natural test knowledge space iff* $\mathcal{K}$ *is a quasi ordinal knowledge space.*

Proof. For any family $\mathcal{F} := \{\dot{K} := (T \cap K)_{T\in\mathcal{T}} : K \in \mathcal{I}\}$ ($\mathcal{I} \subset \mathcal{K}$), it holds

$$\dot{\bigcup}\mathcal{F} \in \dot{\mathcal{K}} \Longleftrightarrow \cup_{K\in\mathcal{I}} K \in \mathcal{K},$$
$$\dot{\bigcap}\mathcal{F} \in \dot{\mathcal{K}} \Longleftrightarrow \cap_{K\in\mathcal{I}} K \in \mathcal{K}.$$

[5] Do not mistake '$\dot{\cup}\cdots$' for the *disjoint* union of sets.

□

Example 1. We consider on $Q := \{a_1, a_2, b_1, b_2\}$ the knowledge structure

$$\mathcal{K} := \Big\{\emptyset, \{a_2\}, \{b_2\}, \{a_2, b_2\}, \{b_1, b_2\}, \{a_2, b_1, b_2\}, \{a_1, a_2, b_2\}, Q\Big\},$$

and the set $\mathcal{T} := \Big\{A := \{a_1, a_2\}, B := \{b_1, b_2\}\Big\}$ of tests in Q. The natural test knowledge structure $\dot{\mathcal{K}}$ associated with $(Q, \mathcal{K})$ and $\mathcal{T}$ is reported in Table 1.

[Table 1]

The natural test knowledge structure $\dot{\mathcal{K}}$ is a quasi ordinal natural test knowledge space since $\mathcal{K}$ is a quasi ordinal knowledge space. □

4.2 l-type test knowledge structures

Sometimes it may only be important to know whether a person is capable of mastering *anything* of a test. For instance, imagine a test which contains exactly the different ways a problem can be solved. If we just want to know whether a person is capable of mastering the problem, it is only important to know whether this person is capable of mastering *at least one* of the ways.

This leads to the concept of a *l-type test knowledge structure.*

Definition 29 *Let $(Q, \mathcal{K})$ be a knowledge structure, and let $\mathcal{T}$ be a set of tests in Q. For any $K \in \mathcal{K}$ and $T \in \mathcal{T}$, let*

$$a_T(K; l) := \begin{cases} 1 & : \quad |T \cap K| \geq 1 \\ 0 & : \quad |T \cap K| = 0 \end{cases}.$$

The set

$$\dot{\mathcal{K}}_l := \left\{ \dot{K}_l := (a_T(K;l))_{T\in\mathcal{T}} : K \in \mathcal{K} \right\}$$

of all $|\mathcal{T}|$*-tuples* $\dot{K}_l := (a_T(K;l))_{T\in\mathcal{T}}$ *(*$K \in \mathcal{K}$*) is called the l-type test knowledge structure associated with* $(Q, \mathcal{K})$ *and* $\mathcal{T}$*. The* $|\mathcal{T}|$*-tuple* $\dot{K}_l := (a_T(K;l))_{T\in\mathcal{T}}$ *is called the l-type test knowledge state corresponding to* $K \in \mathcal{K}$*.*

If $K \in \mathcal{K}$ is the knowledge state of a person, her/his l-type test knowledge state $\dot{K}_l := (a_T(K;l))_{T\in\mathcal{T}}$ contains information about in which of the tests $T \in \mathcal{T}$ this person is capable of mastering *anything* $(a_T(K;l) = 1)$ or *nothing* $(a_T(K;l) = 0)$.

The next corollary is a consequence of Definition 29 and Theorem 8, and it states that the l-type test knowledge structure can be characterized by (derived from) the underlying surmise relation among items.

Corollary 30 *Let* $(Q, \mathcal{K})$ *be a quasi ordinal knowledge space and* $\mathcal{T}$ *a set of tests in* Q*. Let* S *be the surmise relation associated with* $(Q, \mathcal{K})$*. Let* $\dot{\mathcal{K}}_l$ *be the l-type test knowledge structure associated with* $(Q, \mathcal{K})$ *and* $\mathcal{T}$*. Then, for any* $|\mathcal{T}|$*-tuple* $\mathbf{F} := (F_T)_{T\in\mathcal{T}}$ *of zeros and ones* $F_T \in \{0, 1\}$ *(*$T \in \mathcal{T}$*), it holds*

$$\mathbf{F} \in \dot{\mathcal{K}}_l \iff \exists G \subset Q : \Big\{ \big[\forall (p,q) \in S : [q \in G \Longrightarrow p \in G]\big] \text{ and }$$
$$\big[\forall T \in \mathcal{T} : \big([F_T = 0 \Longrightarrow T \cap G = \emptyset] \text{ and } [F_T = 1 \Longrightarrow T \cap G \neq \emptyset]\big)\big] \Big\}.$$

□

Example 2. Let Q, $\mathcal{K}$, and $\mathcal{T}$ be defined as in Example 1. The associated l-type test knowledge structure $\dot{\mathcal{K}}_l$ is reported in Table 2.

[Table 2]

□

4.3 r-type test knowledge structures

Sometimes it may only be important to know whether a person is capable of mastering *everything* of a test. For instance, imagine a test in which each item assesses a different skill, and assume that all the skills assessed by the test are necessary and sufficient for solving a problem. If we just want to know whether a person is capable of mastering the problem, it is only important to know whether this person is capable of mastering *all* items of the test.

This leads to the concept of a *r-type test knowledge structure.*

Definition 31 *Let $(Q, \mathcal{K})$ be a knowledge structure, and let $\mathcal{T}$ be a set of tests in Q. For any $K \in \mathcal{K}$ and $T \in \mathcal{T}$, let*

$$a_T(K; r) := \begin{cases} 1 & : \quad |T \cap K| = |T| \\ 0 & : \quad |T \cap K| < |T| \end{cases}.$$

The set

$$\dot{\mathcal{K}}_r := \left\{ \dot{K}_r := (a_T(K; r))_{T \in \mathcal{T}} : K \in \mathcal{K} \right\}$$

of all $|\mathcal{T}|$-tuples $\dot{K}_r := (a_T(K; r))_{T \in \mathcal{T}}$ ($K \in \mathcal{K}$) is called the r-type test knowledge structure associated with $(Q, \mathcal{K})$ and $\mathcal{T}$. The $|\mathcal{T}|$-tuple $\dot{K}_r := (a_T(K; r))_{T \in \mathcal{T}}$ is called the r-type test knowledge state corresponding to $K \in \mathcal{K}$.

If $K \in \mathcal{K}$ is the knowledge state of a person, her/his r-type test knowledge state $\dot{K}_r := (a_T(K; r))_{T \in \mathcal{T}}$ contains information about in which of the tests

$T \in \mathcal{T}$ this person is capable of mastering *everything* ($a_T(K;r) = 1$) or not ($a_T(K;r) = 0$).

The next corollary is a consequence of Definition 31 and Theorem 8, and it states that the r-type test knowledge structure can be characterized by (derived from) the underlying surmise relation among items.

Corollary 32 *Let* $(Q, \mathcal{K})$ *be a quasi ordinal knowledge space and* $\mathcal{T}$ *a set of tests in* Q*. Let* S *be the surmise relation associated with* $(Q, \mathcal{K})$*. Let* $\dot{\mathcal{K}}_r$ *be the r-type test knowledge structure associated with* $(Q, \mathcal{K})$ *and* $\mathcal{T}$*. Then, for any* $|\mathcal{T}|$*-tuple* $\mathbf{F} := (F_T)_{T \in \mathcal{T}}$ *of zeros and ones* $F_T \in \{0, 1\}$ *(*$T \in \mathcal{T}$*), it holds*

$$\mathbf{F} \in \dot{\mathcal{K}}_r \Longleftrightarrow \exists G \subset Q : \Big\{ \big[\forall (p,q) \in S : [q \in G \Longrightarrow p \in G]\big] \text{ and } \\ \big[\forall T \in \mathcal{T} : \big([F_T = 0 \Longrightarrow T \cap G \neq T] \text{ and } [F_T = 1 \Longrightarrow T \cap G = T]\big)\big]\Big\}.$$

□

Example 3. Let Q, $\mathcal{K}$, and $\mathcal{T}$ be defined as in Example 1. The associated r-type test knowledge structure $\dot{\mathcal{K}}_r$ is reported in Table 3.

[Table 3]

□

4.4 c-type test knowledge structures

In traditional and modern psychological test theories such as the Guttman (1944) scalogram technique, the Rasch (1960) model, and Mokken's (1971) monotone homogeneity model, an examinee is 'characterized' by the number of items she/he has solved in each of the sub-tests of an upper-test. This is called the *test profile* of the examinee, and has a straightforward mathematization in the context of test knowledge structures.

This leads to the concept of a *c-type test knowledge structure.*

Definition 33 *Let* $(Q, \mathcal{K})$ *be a knowledge structure, and let* $\mathcal{T}$ *be a set of tests in* Q*. For any* $K \in \mathcal{K}$ *and* $T \in \mathcal{T}$*, let* $a_T(K; c) := |T \cap K|$*. The set*

$$\dot{\mathcal{K}}_c := \left\{ \dot{K}_c := (a_T(K; c))_{T \in \mathcal{T}} : K \in \mathcal{K} \right\}$$

of all $|\mathcal{T}|$*-tuples* $\dot{K}_c := (a_T(K; c))_{T \in \mathcal{T}}$ $(K \in \mathcal{K})$ *is called the c-type test knowledge structure associated with* $(Q, \mathcal{K})$ *and* $\mathcal{T}$*. The* $|\mathcal{T}|$*-tuple* $\dot{K}_c := (a_T(K; c))_{T \in \mathcal{T}}$ *is called the c-type test knowledge state corresponding to* $K \in \mathcal{K}$*.*

If $K \in \mathcal{K}$ is the knowledge state of a person, her/his c-type test knowledge state contains information about *how many* items in each of the tests this person is capable of mastering.

The next corollary is a consequence of Definition 33 and Theorem 8, and it states that the c-type test knowledge structure can be characterized by (derived from) the underlying surmise relation among items.

Corollary 34 *Let* $(Q, \mathcal{K})$ *be a quasi ordinal knowledge space and* $\mathcal{T}$ *a set of tests in* Q*. Let* S *be the surmise relation associated with* $(Q, \mathcal{K})$*. Let* $\dot{\mathcal{K}}_c$ *be the c-type test knowledge structure associated with* $(Q, \mathcal{K})$ *and* $\mathcal{T}$*. Then, for any* $|\mathcal{T}|$*-tuple* $\mathbf{F} := (F_T)_{T \in \mathcal{T}}$ *of non-negative integers* $F_T \in \{0, 1, 2, \ldots, |T|\}$ $(T \in \mathcal{T})$*, it holds*

$$\mathbf{F} \in \dot{\mathcal{K}}_c \Longleftrightarrow \exists G \subset Q : \left\{ \left[\forall (p, q) \in S : [q \in G \Longrightarrow p \in G] \right] \right.$$

$$\text{and } \Big[\forall T \in \mathcal{T} : |T \cap G| = F_T\Big]\Big\}.$$

□

Example 4. Let Q, $\mathcal{K}$, and $\mathcal{T}$ be defined as in Example 1. The associated c-type test knowledge structure $\dot{\mathcal{K}}_c$ is reported in Table 4.

[Table 4]

□

5 Characterizations of the test surmise relations

5.1 Characterizing the test surmise relations via the knowledge structure and surmise relation

By definition, the natural, left-, right-, and total-covering test surmise relations are characterized by the underlying knowledge structure at the level of items; in respective order, see Definitions 11, 14, 17, and 20. We have further noted that these test surmise relations can also be characterized by the underlying surmise relation among items; in respective order, see Corollaries 12, 15, and 18, and Subsection 3.5.

The following example illustrates the non-uniqueness of the characterizations of the test surmise relations via the knowledge structure and surmise relation. That is, different knowledge structures and different surmise relations can lead to the same test surmise relations, under the characterization formulas described in the afore mentioned definitions and corollaries.

Example 5. Let $Q := \{a_1, a_2, b\}$ and $\mathcal{T} := \Big\{A := \{a_1, a_2\}, B := \{b\}\Big\}$. Consider on Q the two (different) quasi orders S_1 and S_2,

$$S_1 := \triangle_{Q\times Q} \cup \{(a_1, b)\},$$
$$S_2 := \triangle_{Q\times Q} \cup \{(a_2, b)\}.$$

According to the Birkhoff–Theorem (Theorem 8), S_1 and S_2 are the surmise relations associated with two (different) knowledge structures $\mathcal{K}_1$ and $\mathcal{K}_2$ on Q, respectively. The test surmise relations associated with $(Q, \mathcal{K}_1)$ (in the following indexed by 1) and $(Q, \mathcal{K}_2)$ (in the following indexed by 2) and $\mathcal{T}$ are given by

$$\mathrm{NTSR}_1 = \mathrm{NTSR}_2 = \Big\{(A, A), (B, B), (A, B)\Big\},$$
$$\mathrm{LTSR}_1 = \mathrm{LTSR}_2 = \Big\{(A, A), (B, B), (A, B)\Big\},$$
$$\mathrm{RTSR}_1 = \mathrm{RTSR}_2 = \Big\{(A, A), (B, B)\Big\},$$
$$\mathrm{TTSR}_1 = \mathrm{TTSR}_2 = \Big\{(A, A), (B, B)\Big\}.$$

□

5.2 Characterizing the test surmise relations via the test knowledge structures

Next we discuss characterizations of the natural, left-, right-, and total-covering test surmise relations via the natural, l-type, r-type, and c-type test knowledge structures. In the sequel we will use the following notation: For a knowledge structure $(Q, \mathcal{K})$, a set $\mathcal{T}$ of tests in Q, and for any $T \in \mathcal{T}$ and $K \in \mathcal{K}$, let $T_K := T \cap K$.

Characterizing the natural test surmise relations

The characterization of the natural test surmise relation via the natural test knowledge structure becomes especially informative in the case of a quasi ordinal natural test knowledge space. This is the content of the second part of the next proposition.

Proposition 35 *Let $(Q, \mathcal{K})$ be a knowledge structure, and let $\mathcal{T}$ be a set of tests in Q. Let $\dot{\mathcal{S}}$ and $\dot{\mathcal{K}}$ denote the natural test surmise relation and natural test knowledge structure associated with $(Q, \mathcal{K})$ and $\mathcal{T}$, respectively. It holds:*

1. *For any $A, B \in \mathcal{T}$,*

$$B \; \dot{\mathcal{S}} \; A \Longleftrightarrow \Big[\exists a \in A \; \exists b \in B \; \forall \dot{K} = (T_K)_{T \in \mathcal{T}} \in \dot{\mathcal{K}} : \big[a \in A_K \Longrightarrow b \in B_K\big]\Big].$$

2. *If $\dot{\mathcal{K}}$ is a quasi ordinal natural test knowledge space, then, for any $A, B \in \mathcal{T}$,*

$$B \; \dot{\mathcal{S}} \; A \Longleftrightarrow \Big[\forall \dot{K} = (T_K)_{T \in \mathcal{T}} \in \dot{\mathcal{K}} \text{ with } B_K = \emptyset : A_K \neq A\Big].$$

Proof. 1. For any $a \in A$ and $b \in B$, it holds:

$$\begin{aligned} &\forall \dot{K} = (T_K)_{T \in \mathcal{T}} \in \dot{\mathcal{K}} : \big[a \in A_K \Longrightarrow b \in B_K\big] \\ &\quad \Longleftrightarrow \forall K \in \mathcal{K} : \big[a \in K \Longrightarrow b \in K\big] \\ &\quad \Longleftrightarrow b \in \bigcap \mathcal{K}_a. \end{aligned}$$

2. Let $\mathcal{K}' := \{K \in \mathcal{K} : B \cap K = \emptyset\}$ and $\mathcal{K}'' := \mathcal{K} \setminus \mathcal{K}'$. Because $\mathcal{K}$ is a quasi ordinal knowledge space (Corollary 28) and $B \cap \bigcup \mathcal{K}' = \emptyset$, it holds $\bigcup \mathcal{K}' \in \mathcal{K}'$. It also holds:

$$\begin{aligned} &\forall \dot{K} = (T_K)_{T \in \mathcal{T}} \in \dot{\mathcal{K}} \text{ with } B_K = \emptyset : A_K \neq A \\ &\quad \Longleftrightarrow \forall K \in \mathcal{K}' : A \cap K \neq A. \end{aligned}$$

'$\Longrightarrow$:' Let $\hat{K} \in \mathcal{K}'$. Since $B \,\dot{\mathcal{S}}\, A$, there exist some elements $a \in A$ and $b \in B$ such that $b \in K$ for any $K \in \mathcal{K}_a$. Since $B \cap \hat{K} = \emptyset$, it holds $\hat{K} \notin \mathcal{K}_a$. Thus $a \notin \hat{K}$, and $A \cap \hat{K} \neq A$.

'$\Longleftarrow$:' Because $\bigcup \mathcal{K}' \in \mathcal{K}'$ and $A \cap K \neq A$ for any $K \in \mathcal{K}'$, there is an element $a \in A \setminus \bigcup \mathcal{K}'$. Thus $\mathcal{K}_a \subset \mathcal{K}''$. Because $\mathcal{K}$ is a quasi ordinal knowledge space, $\bigcap \mathcal{K}_a \in \mathcal{K}_a \subset \mathcal{K}''$. Therefore $B \cap \bigcap \mathcal{K}_a \neq \emptyset$. □

Example 6. Let Q, $\mathcal{K}$, and $\mathcal{T}$ be defined as in Example 1. The associated natural test knowledge structure $\dot{\mathcal{K}}$ is a quasi ordinal natural test knowledge space. According to the second part of Proposition 35, the associated natural test surmise relation $\dot{\mathcal{S}}$ can be derived from $\dot{\mathcal{K}}$ as described in Tables 5 and 6, which respectively report that $B \,\dot{\mathcal{S}}\, A$ and $A \,\not\!\dot{\mathcal{S}}\, B$ (i.e., not $A \,\dot{\mathcal{S}}\, B$).

[Table 5]

[Table 6]

□

The next corollary is a consequence of Proposition 35, and it states that, in the case of an underlying quasi ordinal knowledge space, the natural test surmise relation can be characterized by (derived from) the associated c-type test knowledge structure.

Corollary 36 *Let $(Q, \mathcal{K})$ be a quasi ordinal knowledge space, and let $\mathcal{T}$ be a set of tests in Q. Let $\dot{\mathcal{S}}$ and $\dot{\mathcal{K}}_c$ denote the natural test surmise relation and c-type test knowledge structure associated with $(Q, \mathcal{K})$ and $\mathcal{T}$, respectively. Then, for any $A, B \in \mathcal{T}$,*

$$B \,\dot{\mathcal{S}}\, A \Longleftrightarrow \Big[\forall \dot{K}_c = (a_T(K;c))_{T \in \mathcal{T}} \in \dot{\mathcal{K}}_c : \big[a_B(K;c) = 0 \Longrightarrow a_A(K;c) < |A|\big]\Big].$$

□

Characterizing the left-covering test surmise relations

Proposition 37 *Let $(Q, \mathcal{K})$ be a knowledge structure, and let $\mathcal{T}$ be a set of tests in Q. Let $\dot{\mathcal{S}}_l$, $\dot{\mathcal{K}}$, and $\dot{\mathcal{K}}_l$ denote the left-covering test surmise relation, natural test knowledge structure, and l-type test knowledge structure associated with $(Q, \mathcal{K})$ and $\mathcal{T}$, respectively. It holds:*

1. *For any $A, B \in \mathcal{T}$,*

$$B\ \dot{\mathcal{S}}_l\ A \Longleftrightarrow \Big[\forall a \in A\ \exists b \in B\ \forall \dot{K} = (T_K)_{T\in\mathcal{T}} \in \dot{\mathcal{K}} : \big[a \in A_K \Longrightarrow b \in B_K\big]\Big].$$

2. *If $\dot{\mathcal{K}}$ is a quasi ordinal natural test knowledge space, then, for any $A, B \in \mathcal{T}$,*

$$B\ \dot{\mathcal{S}}_l\ A \Longleftrightarrow \Big[\forall \dot{K}_l = (a_T(K;l))_{T\in\mathcal{T}} \in \dot{\mathcal{K}}_l \text{ with } a_B(K;l) = 0 : a_A(K;l) = 0\Big].$$

Proof. 1. See the proof of the first part of Proposition 35.
2. Let $\mathcal{K}' := \{K \in \mathcal{K} : B \cap K = \emptyset\}$ and $\mathcal{K}'' := \mathcal{K} \setminus \mathcal{K}'$. It holds:

$$\begin{aligned}&\forall \dot{K}_l = (a_T(K;l))_{T\in\mathcal{T}} \in \dot{\mathcal{K}}_l \text{ with } a_B(K;l) = 0 : a_A(K;l) = 0\\ &\quad\Longleftrightarrow \forall K \in \mathcal{K}' : A \cap K = \emptyset.\end{aligned}$$

'$\Longrightarrow$:' Let $\hat{K} \in \mathcal{K}'$. Since $B\ \dot{\mathcal{S}}_l\ A$, there exists, for any $a \in A$, an element $b = b(a) \in B$ such that $b \in K$ for any $K \in \mathcal{K}_a$. Since $B \cap \hat{K} = \emptyset$, it holds $\hat{K} \notin \mathcal{K}_a$ for any $a \in A$. Thus $a \notin \hat{K}$ for any $a \in A$, and $A \cap \hat{K} = \emptyset$.
'$\Longleftarrow$:' Because $A \cap K = \emptyset$ for any $K \in \mathcal{K}'$, it holds $\mathcal{K}_a \subset \mathcal{K}''$ for any $a \in A$. Then $\bigcap \mathcal{K}_a \in \mathcal{K}_a \subset \mathcal{K}''$ for any $a \in A$, and $B \cap \bigcap \mathcal{K}_a \neq \emptyset$ for any $a \in A$. □

Example 7. We continue with Example 2. According to the second part of Proposition 37, the associated left-covering test surmise relation $\dot{\mathcal{S}}_l$ can be derived from $\dot{\mathcal{K}}_l$ as described in Tables 7 and 8, which respectively report that $B \mathrel{\dot{\not\mathcal{S}}_l} A$ and $A \mathrel{\dot{\not\mathcal{S}}_l} B$.

[Table 7]

[Table 8]

□

The next corollary is a consequence of Proposition 37, and it states that, in the case of an underlying quasi ordinal knowledge space, the left-covering test surmise relation can be characterized by (derived from) the associated natural test knowledge structure and c-type test knowledge structure.

Corollary 38 *Let $(Q, \mathcal{K})$ be a quasi ordinal knowledge space, and let $\mathcal{T}$ be a set of tests in Q. Let $\dot{\mathcal{S}}_l$, $\dot{\mathcal{K}}$, and $\dot{\mathcal{K}}_c$ denote the left-covering test surmise relation, natural test knowledge structure, and c-type test knowledge structure associated with $(Q, \mathcal{K})$ and $\mathcal{T}$, respectively. Then, for any $A, B \in \mathcal{T}$, the following statements are equivalent:*

1. $B \mathrel{\dot{\mathcal{S}}_l} A$;
2. $\forall \dot{K} = (T_K)_{T \in \mathcal{T}} \in \dot{\mathcal{K}} : \Big[B_K = \emptyset \Longrightarrow A_K = \emptyset\Big]$;
3. $\forall \dot{K}_c = (a_T(K; c))_{T \in \mathcal{T}} \in \dot{\mathcal{K}}_c : \Big[a_B(K; c) = 0 \Longrightarrow a_A(K; c) = 0\Big]$.

□

Characterizing the right-covering test surmise relations

Proposition 39 *Let $(Q, \mathcal{K})$ be a knowledge structure, and let $\mathcal{T}$ be a set of tests in Q. Let $\dot{\mathcal{S}}_r$, $\dot{\mathcal{K}}$, and $\dot{\mathcal{K}}_r$ denote the right-covering test surmise relation, natural test knowledge structure, and r-type test knowledge structure associated* *with $(Q, \mathcal{K})$ and $\mathcal{T}$, respectively. It holds:*

1. *For any $A, B \in \mathcal{T}$,*

$$B \ \dot{\mathcal{S}}_r \ A \Longleftrightarrow \Big[\forall b \in B \ \exists a \in A \ \forall \dot{K} = (T_K)_{T \in \mathcal{T}} \in \dot{\mathcal{K}} : \big[a \in A_K \Longrightarrow b \in B_K\big]\Big].$$

2. *If $\dot{\mathcal{K}}$ is a quasi ordinal natural test knowledge space, then, for any $A, B \in \mathcal{T}$,*

$$B \ \dot{\mathcal{S}}_r \ A \Longleftrightarrow \Big[\forall \dot{K}_r = (a_T(K;r))_{T \in \mathcal{T}} \in \dot{\mathcal{K}}_r \text{ with } a_A(K;r) = 1 : a_B(K;r) = 1\Big].$$

Proof. 1. See the proof of the first part of Proposition 35.

2. It holds:

$$\begin{aligned} &\forall \dot{K}_r = (a_T(K;r))_{T \in \mathcal{T}} \in \dot{\mathcal{K}}_r \text{ with } a_A(K;r) = 1 : a_B(K;r) = 1 \\ &\quad \Longleftrightarrow \forall K \in \mathcal{K} : \big[A \cap K = A \Longrightarrow B \cap K = B\big]. \end{aligned}$$

'$\Longrightarrow$:' Let $\hat{K} \in \mathcal{K}$ with $A \cap \hat{K} = A$. Let $b \in B$. Since $B \ \dot{\mathcal{S}}_r \ A$, there exists an element $a \in A$ such that $b \in K$ for any $K \in \mathcal{K}_a$. Since $\hat{K} \in \mathcal{K}_a$, it holds $b \in B \cap \hat{K}$.

'$\Longleftarrow$:' Under the assumption that $B \ \dot{\not{\mathcal{S}}}_r \ A$, there exists an element $b \in B$ such that $b \notin \bigcap \mathcal{K}_a$ for any $a \in A$. Using $\hat{K} := \bigcup \mathcal{K}'$ where $\mathcal{K}' := \{\bigcap \mathcal{K}_a : a \in A\}$, then it holds $\hat{K} \in \mathcal{K}$, $b \notin \hat{K}$, and $A \cap \hat{K} = A$. □

Example 8. We continue with Example 3. According to the second part of Proposition 39, the associated right-covering test surmise relation $\dot{\mathcal{S}}_r$ can be derived from $\dot{\mathcal{K}}_r$ as described in Tables 9 and 10, which respectively report that $B \ \dot{\not{\mathcal{S}}}_r \ A$ and $A \ \dot{\not{\mathcal{S}}}_r \ B$.

[Table 9]

[Table 10]

□

The next corollary is a consequence of Proposition 39, and it states that, in the case of an underlying quasi ordinal knowledge space, the right-covering test surmise relation can be characterized by (derived from) the associated natural test knowledge structure and c-type test knowledge structure.

Corollary 40 *Let $(Q, \mathcal{K})$ be a quasi ordinal knowledge space, and let $\mathcal{T}$ be a set of tests in Q. Let $\dot{\mathcal{S}}_r$, $\dot{\mathcal{K}}$, and $\dot{\mathcal{K}}_c$ denote the right-covering test surmise relation, natural test knowledge structure, and c-type test knowledge structure associated with $(Q, \mathcal{K})$ and $\mathcal{T}$, respectively. Then, for any $A, B \in \mathcal{T}$, the following statements are equivalent:*

1. $B \; \dot{\mathcal{S}}_r \; A$;
2. $\forall \dot{K} = (T_K)_{T \in \mathcal{T}} \in \dot{\mathcal{K}} : \Big[A_K = A \Longrightarrow B_K = B \Big]$;
3. $\forall \dot{K}_c = (a_T(K; c))_{T \in \mathcal{T}} \in \dot{\mathcal{K}}_c : \Big[a_A(K; c) = |A| \Longrightarrow a_B(K; c) = |B| \Big]$.

□

Characterizing the total-covering test surmise relations

Let $(Q, \mathcal{K})$ be a knowledge structure and $\mathcal{T}$ a set of tests in Q. Because the total-covering test surmise relation $\dot{\mathcal{S}}_t$ associated with $(Q, \mathcal{K})$ and $\mathcal{T}$ is the intersection of the associated left- and right-covering test surmise relations $\dot{\mathcal{S}}_l$ and $\dot{\mathcal{S}}_r$, a number of characterizations of $\dot{\mathcal{S}}_t$ can be obtained by combining the previously described characterizations of $\dot{\mathcal{S}}_l$ and $\dot{\mathcal{S}}_r$.

For instance, if $\mathcal{K}$ is a quasi ordinal knowledge space, $B \; \dot{\mathcal{S}}_t \; A$ if, and only if,

$$\forall \dot{K} = (T_K)_{T \in \mathcal{T}} \in \dot{\mathcal{K}} : \Big[\Big[B_K = \emptyset \Longrightarrow A_K = \emptyset \Big] \text{ and } \Big[A_K = A \Longrightarrow B_K = B \Big] \Big].$$

This is a characterization of the total-covering test surmise relation via the quasi ordinal natural test knowledge space. Of course, 'mixed' combinations of the characterizations of the left- and right-covering test surmise relations are possible as well. For instance, if $\mathcal{K}$ is a quasi ordinal knowledge space, $B\ \dot{\mathcal{S}}_t\ A$ if, and only if,

$$\forall \dot{K}_l = (a_T(K;l))_{T\in\mathcal{T}} \in \dot{\mathcal{K}}_l : \Big[a_B(K;l) = 0 \Longrightarrow a_A(K;l) = 0\Big]$$

and

$$\forall \dot{K}_c = (a_T(K;c))_{T\in\mathcal{T}} \in \dot{\mathcal{K}}_c : \Big[a_A(K;c) = |A| \Longrightarrow a_B(K;c) = |B|\Big].$$

This is a characterization of the total-covering test surmise relation via the l- and c-type test knowledge structures associated with an underlying quasi ordinal knowledge space.

5.3 Non-uniqueness of the characterizations of the test surmise relations

The next two examples illustrate that the characterizations of the test surmise relations via the test knowledge structures discussed in Subsection 5.2 are not unique in general.

Example 9. Let $Q := \{a_1, a_2, b\}$ and $\mathcal{T} := \Big\{A := \{a_1, a_2\}, B := \{b\}\Big\}$. Consider on Q the quasi orders S_1 and S_2,

$$S_1 := \triangle_{Q\times Q} \cup \{(a_1, b), (a_2, b)\},$$
$$S_2 := \triangle_{Q\times Q} \cup \{(a_2, a_1), (a_2, b), (a_1, b)\}.$$

They are the surmise relations associated with the quasi ordinal knowledge spaces, in respective order,

$$\mathcal{K}_1 = \Big\{\emptyset, \{a_1\}, \{a_2\}, \{a_1, b\}, \{a_2, b\}, Q\Big\},$$
$$\mathcal{K}_2 = \Big\{\emptyset, \{a_2\}, \{a_1, a_2\}, Q\Big\}.$$

The natural, l-type, r-type, and c-type test knowledge structures associated with $(Q, \mathcal{K}_1)$ (in the following indexed by 1) and $(Q, \mathcal{K}_2)$ (in the following indexed by 2) and $\mathcal{T}$ are, in respective order,

$$\dot{\mathcal{K}}_1 = \Big\{(\emptyset, \emptyset), (\{a_1\}, \emptyset), (\{a_2\}, \emptyset), (\{a_1\}, \{b\}), (\{a_2\}, \{b\}), (A, B)\Big\},$$
$$\dot{\mathcal{K}}_2 = \Big\{(\emptyset, \emptyset), (\{a_2\}, \emptyset), (A, \emptyset), (A, B)\Big\},$$
$$\dot{\mathcal{K}}_{1l} = \Big\{(0,0), (1,0), (1,1)\Big\},$$
$$\dot{\mathcal{K}}_{2l} = \Big\{(0,0), (1,0), (1,1)\Big\},$$
$$\dot{\mathcal{K}}_{1r} = \Big\{(0,0), (0,1), (1,1)\Big\},$$
$$\dot{\mathcal{K}}_{2r} = \Big\{(0,0), (1,0), (1,1)\Big\},$$
$$\dot{\mathcal{K}}_{1c} = \Big\{(0,0), (1,0), (1,1), (2,1)\Big\},$$
$$\dot{\mathcal{K}}_{2c} = \Big\{(0,0), (1,0), (2,0), (2,1)\Big\}.$$

The natural, left-, right-, and total-covering test surmise relations associated with $(Q, \mathcal{K}_1)$ and $(Q, \mathcal{K}_2)$ and $\mathcal{T}$ are all identical, equal to $\Big\{(A, A), (B, B), (A, B)\Big\}$; that is,

$$\dot{\mathcal{S}}_1 = \dot{\mathcal{S}}_2 = \dot{\mathcal{S}}_{\theta\mu} = \Big\{(A, A), (B, B), (A, B)\Big\} \qquad (\text{for } \theta \in \{1, 2\},\ \mu \in \{l, r, t\}).$$

Now one can make the following observations:[6]

Natural test surmise relation. According to Proposition 35 and Corollary 36, the natural test surmise relations $\dot{\mathcal{S}}_1$ and $\dot{\mathcal{S}}_2$ are characterized by $\dot{\mathcal{K}}_1$, $\dot{\mathcal{K}}_{1c}$ and $\dot{\mathcal{K}}_2$, $\dot{\mathcal{K}}_{2c}$, respectively. However, $\dot{\mathcal{S}}_1 = \dot{\mathcal{S}}_2$, and $\dot{\mathcal{K}}_1 \neq \dot{\mathcal{K}}_2$ and $\dot{\mathcal{K}}_{1c} \neq \dot{\mathcal{K}}_{2c}$.

[6] The fact that the test surmise relations may not be uniquely characterized by the underlying knowledge structures and surmise relations has been already discussed in Subsection 5.1.

Left-covering test surmise relation. According to Proposition 37 and Corollary 38, the left-covering test surmise relations $\dot{\mathcal{S}}_{1l}$ and $\dot{\mathcal{S}}_{2l}$ are characterized by $\dot{\mathcal{K}}_1$, $\dot{\mathcal{K}}_{1l}$, $\dot{\mathcal{K}}_{1c}$ and $\dot{\mathcal{K}}_2$, $\dot{\mathcal{K}}_{2l}$, $\dot{\mathcal{K}}_{2c}$, respectively. However, $\dot{\mathcal{S}}_{1l} = \dot{\mathcal{S}}_{2l}$, and $\dot{\mathcal{K}}_1 \neq \dot{\mathcal{K}}_2$ and $\dot{\mathcal{K}}_{1c} \neq \dot{\mathcal{K}}_{2c}$. But $\dot{\mathcal{K}}_{1l} = \dot{\mathcal{K}}_{2l}$; that the characterization of the left-covering test surmise relation via the l-type test knowledge structure may not be unique as well, will be seen below in Example 10.

Right-covering test surmise relation. According to Proposition 39 and Corollary 40, the right-covering test surmise relations $\dot{\mathcal{S}}_{1r}$ and $\dot{\mathcal{S}}_{2r}$ are characterized by $\dot{\mathcal{K}}_1$, $\dot{\mathcal{K}}_{1r}$, $\dot{\mathcal{K}}_{1c}$ and $\dot{\mathcal{K}}_2$, $\dot{\mathcal{K}}_{2r}$, $\dot{\mathcal{K}}_{2c}$, respectively. However, $\dot{\mathcal{S}}_{1r} = \dot{\mathcal{S}}_{2r}$, and $\dot{\mathcal{K}}_1 \neq \dot{\mathcal{K}}_2$, $\dot{\mathcal{K}}_{1r} \neq \dot{\mathcal{K}}_{2r}$, and $\dot{\mathcal{K}}_{1c} \neq \dot{\mathcal{K}}_{2c}$.

Total-covering test surmise relation. It is obvious that the characterizations of the total-covering test surmise relations $\dot{\mathcal{S}}_{1t}$ and $\dot{\mathcal{S}}_{2t}$ via the test knowledge structures, derived as combinations of the characterizations of the left- and right-covering test surmise relations $\dot{\mathcal{S}}_{1l}$, $\dot{\mathcal{S}}_{2l}$ and $\dot{\mathcal{S}}_{1r}$, $\dot{\mathcal{S}}_{2r}$, respectively, are not unique. For instance, $\dot{\mathcal{S}}_t := \dot{\mathcal{S}}_{1t} = \dot{\mathcal{S}}_{2t}$ can be characterized by both $\dot{\mathcal{K}}_1$ and $\dot{\mathcal{K}}_2$, $\dot{\mathcal{K}}_1 \neq \dot{\mathcal{K}}_2$, through: For any $C, D \in \mathcal{T}$, $D \, \dot{\mathcal{S}}_t \, C$ if, and only if,

$$\forall \dot{K} = (T_K)_{T \in \mathcal{T}} \in \dot{\mathcal{K}}_i : \Big[\big[D_K = \emptyset \Longrightarrow C_K = \emptyset\big] \text{ and } \big[C_K = C \Longrightarrow D_K = D\big]\Big]$$

$(i = 1, 2)$.

□

The characterization of the left-covering test surmise relation via the l-type test knowledge structure may not be unique as well.

Example 10. Let $Q := \{a, b, c_1, c_2, d\}$ and $\mathcal{T} := \Big\{A := \{a\}, B := \{b\}, C := \{c_1, c_2\}, D := \{d\}\Big\}$. Consider on Q the quasi orders S_1 and S_2,

$$S_1 := \triangle_{Q\times Q} \cup \{(a, b), (a, c_1), (d, c_2)\},$$

$$S_2 := \triangle_{Q\times Q} \cup \{(a,b),(d,c_2)\}.$$

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ denote the corresponding quasi ordinal knowledge spaces. The left-covering test surmise relations $\dot{\mathcal{S}}_{1l}$ and $\dot{\mathcal{S}}_{2l}$ respectively associated with $(Q,\mathcal{K}_1)$ and $(Q,\mathcal{K}_2)$ and $\mathcal{T}$ are identical,

$$\dot{\mathcal{S}}_{1l} = \dot{\mathcal{S}}_{2l} = \Big\{(A,A),(B,B),(C,C),(D,D),(A,B)\Big\}.$$

The l-type test knowledge structures $\dot{\mathcal{K}}_{1l}$ and $\dot{\mathcal{K}}_{2l}$ respectively associated with $(Q,\mathcal{K}_1)$ and $(Q,\mathcal{K}_2)$ and $\mathcal{T}$, however, are different. For instance, the 4-tuple $\mathbf{F} := (0,0,1,0)$ belongs to $\dot{\mathcal{K}}_{2l}$, but does not belong to $\dot{\mathcal{K}}_{1l}$. This is because $\{c_1\} \in \mathcal{K}_2$, and it holds:

$$\begin{aligned}\dot{\mathcal{K}}_{2l} &\ni \dot{\{c_1\}}_l \\ &:= (a_A(\{c_1\};l), a_B(\{c_1\};l), a_C(\{c_1\};l), a_D(\{c_1\};l)) \\ &= (0,0,1,0).\end{aligned}$$

Under the assumption that $\mathbf{F} \in \dot{\mathcal{K}}_{1l}$, there exists a knowledge state $K \in \mathcal{K}_1$ such that $A \cap K = \emptyset$, $B \cap K = \emptyset$, $D \cap K = \emptyset$, and $C \cap K \neq \emptyset$. Then $K \neq \emptyset$, and because $\mathcal{T}$ is a partition of Q, it holds $K \subset C$. This, however, contradicts the fact that any knowledge state in $\mathcal{K}_1$ containing an element of C must necessarily contain $a \in A$ or $d \in D$ (or both the elements). □

6 Characterizations of the test knowledge structures

6.1 Characterizing the test knowledge structures via the knowledge structure and surmise relation

By definition, the natural, l-, r-, and c-type test knowledge structures are characterized by the underlying knowledge structure at the level of items; in respective order, see Definitions 24, 29, 31, and 33. According to Theorem 8, these test knowledge structures can also be characterized by the underlying surmise relation among items, associated with a quasi ordinal knowledge space; in respective order, see Corollaries 25, 30, 32, and 34.

The characterizations of the natural test knowledge structure via the knowledge structure and surmise relation (of a quasi ordinal knowledge space) are unique. (It is obvious that different knowledge structures and different surmise relations always lead to different natural test knowledge structures, under the characterization formulas described in Definition 24 and Corollary 25, respectively. This is because the tests form a partition of the domain, and Theorem 8 establishes a one-to-one correspondence.) For the l-, r-, and c-type test knowledge structures, however, the characterizations via the knowledge structure and surmise relation are not unique in general, as illustrated by the following example.

Example 11. Let $Q := \{a_1, a_2, b\}$ and $\mathcal{T} := \Big\{A := \{a_1, a_2\}, B := \{b\}\Big\}$. Consider on Q the quasi orders S_1 and S_2,

$$S_1 := \triangle_{Q\times Q} \cup \{(a_1, b)\},$$
$$S_2 := \triangle_{Q\times Q} \cup \{(a_2, b)\}.$$

They are the surmise relations associated with the quasi ordinal knowledge spaces, in respective order,

$$\mathcal{K}_1 = \Big\{\emptyset, \{a_1\}, \{a_2\}, \{a_1, a_2\}, \{a_1, b\}, Q\Big\},$$
$$\mathcal{K}_2 = \Big\{\emptyset, \{a_1\}, \{a_2\}, \{a_1, a_2\}, \{a_2, b\}, Q\Big\}.$$

The l-, r-, and c-type test knowledge structures associated with $(Q, \mathcal{K}_1)$ (in the following indexed by 1) and $(Q, \mathcal{K}_2)$ (in the following indexed by 2) and $\mathcal{T}$ are, in respective order,

$$\dot{\mathcal{K}}_{1l} = \dot{\mathcal{K}}_{2l} = \Big\{(0,0), (1,0), (1,1)\Big\},$$
$$\dot{\mathcal{K}}_{1r} = \dot{\mathcal{K}}_{2r} = \Big\{(0,0), (1,0), (0,1), (1,1)\Big\},$$
$$\dot{\mathcal{K}}_{1c} = \dot{\mathcal{K}}_{2c} = \Big\{(0,0), (1,0), (1,1), (2,0), (2,1)\Big\}.$$

□

6.2 Characterizing the test knowledge structures via the test surmise relations

Contrary to the test surmise relations, which can be characterized by the test knowledge structures, the test knowledge structures are not necessarily inferable from the test surmise relations. A condition, however, can be proposed, under which the test knowledge structures can be derived from the test surmise relations.

The next corollary is a consequence of Propositions 35, 37, and 39. It addresses the condition for the natural test knowledge structure, in the case of an underlying general knowledge structure.

Corollary 41 *Let $(Q, \mathcal{K})$ be a knowledge structure, and let $\mathcal{T}$ be a set of tests in Q. Let $\dot{\mathcal{K}}$, $\dot{\mathcal{S}}$, $\dot{\mathcal{S}}_l$, $\dot{\mathcal{S}}_r$, and $\dot{\mathcal{S}}_t$ denote the natural test knowledge structure,* *natural test surmise relation, left-covering test surmise relation, right-covering test surmise relation, and total-covering test surmise relation associated with* $(Q, \mathcal{K})$ *and* $\mathcal{T}$*, respectively. Let* $\dot{\mathcal{K}}^*$*,* $\dot{\mathcal{K}}^{*,l}$*,* $\dot{\mathcal{K}}^{*,r}$*, and* $\dot{\mathcal{K}}^{*,t}$ *be defined as follows: For any* $|\mathcal{T}|$*-tuple* $\mathbf{F} := (F_T)_{T\in\mathcal{T}}$ *of subsets* $F_T \subset T$ $(T \in \mathcal{T})$*,*

$$\begin{aligned}
\mathbf{F} \in \dot{\mathcal{K}}^* &:\Longleftrightarrow \Big[\forall (B, A) \in \dot{\mathcal{S}}\ \exists a \in A\ \exists b \in B : \big[a \in F_A \Longrightarrow b \in F_B\big]\Big],\\
\mathbf{F} \in \dot{\mathcal{K}}^{*,l} &:\Longleftrightarrow \Big[\forall (B, A) \in \dot{\mathcal{S}}_l\ \forall a \in A\ \exists b \in B : \big[a \in F_A \Longrightarrow b \in F_B\big]\Big],\\
\mathbf{F} \in \dot{\mathcal{K}}^{*,r} &:\Longleftrightarrow \Big[\forall (B, A) \in \dot{\mathcal{S}}_r\ \forall b \in B\ \exists a \in A : \big[a \in F_A \Longrightarrow b \in F_B\big]\Big],\\
\mathbf{F} \in \dot{\mathcal{K}}^{*,t} &:\Longleftrightarrow \Big\{\Big[\forall (B, A) \in \dot{\mathcal{S}}_t\ \forall a \in A\ \exists b \in B : \big[a \in F_A \Longrightarrow b \in F_B\big]\Big]\\
&\qquad \text{and } \Big[\forall (B, A) \in \dot{\mathcal{S}}_t\ \forall b \in B\ \exists a \in A : \big[a \in F_A \Longrightarrow b \in F_B\big]\Big]\Big\}.
\end{aligned}$$

Then it holds: $\dot{\mathcal{K}}$ *is a subset of* $\dot{\mathcal{K}}^*$*,* $\dot{\mathcal{K}}^{*,l}$*,* $\dot{\mathcal{K}}^{*,r}$*, and* $\dot{\mathcal{K}}^{*,t}$*. In general,* $\dot{\mathcal{K}}$ *is a proper subset of these sets. If* $\dot{\mathcal{K}} = \dot{\mathcal{K}}^*$*,* $\dot{\mathcal{K}} = \dot{\mathcal{K}}^{*,l}$*,* $\dot{\mathcal{K}} = \dot{\mathcal{K}}^{*,r}$*, or* $\dot{\mathcal{K}} = \dot{\mathcal{K}}^{*,t}$*, then* $\dot{\mathcal{K}}$ *can be characterized by (derived from)* $\dot{\mathcal{S}}$*,* $\dot{\mathcal{S}}_l$*,* $\dot{\mathcal{S}}_r$*, or* $\dot{\mathcal{S}}_t$ *using the afore mentioned characterization formulas, respectively.*

Proof. Let $\dot{K} = (T_K)_{T\in\mathcal{T}} \in \dot{\mathcal{K}}$ $(K \in \mathcal{K})$ and $B\ \dot{\mathcal{S}}\ A$ $(A, B \in \mathcal{T})$. According to Proposition 35, there exist some elements $a \in A$ and $b \in B$ such that $a \in A_K$ implies $b \in B_K$. Similarly, one can see that $\dot{\mathcal{K}} \subset \dot{\mathcal{K}}^{*,l}$ (Proposition 37), $\dot{\mathcal{K}} \subset \dot{\mathcal{K}}^{*,r}$ (Proposition 39), and hence $\dot{\mathcal{K}} \subset \dot{\mathcal{K}}^{*,t}$. To show that $\dot{\mathcal{K}}$ is a proper subset of these sets, let $Q := \{a, b_1, b_2\}$, $\mathcal{T} := \big\{A := \{a\}, B := \{b_1, b_2\}\big\}$, and $\mathcal{K} := \big\{\emptyset, \{b_2\}, \{a, b_2\}, \{b_1, b_2\}, Q\big\}$. Then

$$\begin{aligned}
\dot{\mathcal{S}} = \dot{\mathcal{S}}_l &= \big\{(A, A), (B, B), (B, A)\big\},\\
\dot{\mathcal{S}}_r = \dot{\mathcal{S}}_t &= \big\{(A, A), (B, B)\big\},\\
\dot{\mathcal{K}} &= \big\{(\emptyset, \emptyset), (\emptyset, \{b_2\}), (A, \{b_2\}), (\emptyset, B), (A, B)\big\}.
\end{aligned}$$

The 2-tuple $\mathbf{F} := (\emptyset, \{b_1\})$ belongs to any of the sets $\dot{\mathcal{K}}^*$, $\dot{\mathcal{K}}^{*,l}$, $\dot{\mathcal{K}}^{*,r}$, and $\dot{\mathcal{K}}^{*,t}$, but it does not belong to $\dot{\mathcal{K}}$. □

The next corollary is a consequence of Proposition 35 and Corollaries 38 and 40. It addresses the condition for the natural test knowledge structure, in the case of an underlying quasi ordinal knowledge space.

Corollary 42 *Let $(Q, \mathcal{K})$ be a quasi ordinal knowledge space, and let $\mathcal{T}$ be a set of tests in Q. Let $\dot{\mathcal{K}}$, $\dot{\mathcal{S}}$, $\dot{\mathcal{S}}_l$, $\dot{\mathcal{S}}_r$, and $\dot{\mathcal{S}}_t$ denote the natural test knowledge structure, natural test surmise relation, left-covering test surmise relation, right-covering test surmise relation, and total-covering test surmise relation associated with $(Q, \mathcal{K})$ and $\mathcal{T}$, respectively. Let $\dot{\mathcal{K}}^*$, $\dot{\mathcal{K}}^{*,l}$, $\dot{\mathcal{K}}^{*,r}$, and $\dot{\mathcal{K}}^{*,t}$ be defined as follows: For any $|\mathcal{T}|$-tuple $\mathbf{F} := (F_T)_{T\in\mathcal{T}}$ of subsets $F_T \subset T$ $(T \in \mathcal{T})$,*

$$\mathbf{F} \in \dot{\mathcal{K}}^* :\Longleftrightarrow \Big[\forall (B, A) \in \dot{\mathcal{S}} : \big[F_B = \emptyset \Longrightarrow F_A \neq A\big]\Big],$$
$$\mathbf{F} \in \dot{\mathcal{K}}^{*,l} :\Longleftrightarrow \Big[\forall (B, A) \in \dot{\mathcal{S}}_l : \big[F_B = \emptyset \Longrightarrow F_A = \emptyset\big]\Big],$$
$$\mathbf{F} \in \dot{\mathcal{K}}^{*,r} :\Longleftrightarrow \Big[\forall (B, A) \in \dot{\mathcal{S}}_r : \big[F_A = A \Longrightarrow F_B = B\big]\Big],$$
$$\mathbf{F} \in \dot{\mathcal{K}}^{*,t} :\Longleftrightarrow \Big[\forall (B, A) \in \dot{\mathcal{S}}_t : \Big\{\big[F_B = \emptyset \Longrightarrow F_A = \emptyset\big] \text{ and } \big[F_A = A \Longrightarrow F_B = B\big]\Big\}\Big].$$

Then it holds: $\dot{\mathcal{K}}$ is a subset of $\dot{\mathcal{K}}^*$, $\dot{\mathcal{K}}^{*,l}$, $\dot{\mathcal{K}}^{*,r}$, and $\dot{\mathcal{K}}^{*,t}$. In general, $\dot{\mathcal{K}}$ is a proper subset of these sets. If $\dot{\mathcal{K}} = \dot{\mathcal{K}}^*$, $\dot{\mathcal{K}} = \dot{\mathcal{K}}^{*,l}$, $\dot{\mathcal{K}} = \dot{\mathcal{K}}^{*,r}$, or $\dot{\mathcal{K}} = \dot{\mathcal{K}}^{*,t}$, then $\dot{\mathcal{K}}$ can be characterized by (derived from) $\dot{\mathcal{S}}$, $\dot{\mathcal{S}}_l$, $\dot{\mathcal{S}}_r$, or $\dot{\mathcal{S}}_t$ using the afore mentioned characterization formulas, respectively. □

In the same manner as has been done for the natural test knowledge structure, one can obtain some characterizations of the l-, r-, and c-type test knowledge structures via the test surmise relations as well. For instance, if $(Q, \mathcal{K})$ is a quasi ordinal knowledge space, $\mathcal{T}$ a set of tests in Q, and $\dot{\mathcal{S}}$ the natural test surmise relation associated with $(Q, \mathcal{K})$ and $\mathcal{T}$, the associated c-type test knowledge structure $\dot{\mathcal{K}}_c$ is a subset of the set defined by (for any $|\mathcal{T}|$-tuple $\mathbf{F} := (F_T)_{T\in\mathcal{T}}$ of non-negative integers $F_T \in \{0, 1, 2, \ldots, |T|\}$ $(T \in \mathcal{T})$)

$$\mathbf{F} \in \dot{\mathcal{K}}_c^* :\Longleftrightarrow \Big[\forall (B, A) \in \dot{\mathcal{S}} : \big[F_B = 0 \Longrightarrow F_A < |A|\big]\Big].$$

This follows from Corollary 36. In particular, if $\dot{\mathcal{K}}_c = \dot{\mathcal{K}}_c^*$, the c-type test knowledge structure $\dot{\mathcal{K}}_c$ can be characterized by (derived from) the natural test surmise relation $\dot{\mathcal{S}}$ using this characterization formula. Or, if $\dot{\mathcal{S}}_r$ denotes the right-covering test surmise relation associated with $(Q, \mathcal{K})$ and $\mathcal{T}$, the c-type test knowledge structure $\dot{\mathcal{K}}_c$ is a subset of the set defined by (for any $|\mathcal{T}|$-tuple $\mathbf{F} := (F_T)_{T\in\mathcal{T}}$ of non-negative integers $F_T \in \{0, 1, 2, \ldots, |T|\}$ $(T \in \mathcal{T})$)

$$\mathbf{F} \in \dot{\mathcal{K}}_c^{*,r} :\Longleftrightarrow \Big[\forall (B, A) \in \dot{\mathcal{S}}_r : \big[F_A = |A| \Longrightarrow F_B = |B|\big]\Big].$$

This follows from Corollary 40. In particular, if $\dot{\mathcal{K}}_c = \dot{\mathcal{K}}_c^{*,r}$, the c-type test knowledge structure $\dot{\mathcal{K}}_c$ can be characterized by (derived from) the right-covering test surmise relation $\dot{\mathcal{S}}_r$ using this characterization formula.

In this way, a number of at least 'constraint' characterizations of the natural, l-, r-, and c-type test knowledge structures via the natural, left-, right-, and total-covering test surmise relations are obtained, by appropriately modifying and combining the previously described 'unconstraint' characterizations of the test surmise relations.

7 Discussion

7.1 Summary

This paper has examined some possible relationships among 'tests' of items in a knowledge structure, with a test being an element of a partition of the domain of the knowledge structure. The study of tests has been motivated by 'curriculum development' and 'computer-based adaptive assessment and training' (Section 1). Test surmise relations (Section 3) and test knowledge structures (Section 4), based on the underlying surmise relation and knowledge structure at the level of individual items (Section 2), have been investigated. In particular, it has been shown that (a) discriminativity of the underlying knowledge structure on a finite domain implies the property of antisymmetry for the left-, right-, and total-covering test surmise relations, (b) if the domain is not finite, this does not hold in general, and (c) the natural test surmise relations may not necessarily be antisymmetric, even in the case of a discriminative knowledge structure on a finite domain. As the main thrust of this paper, a number of characterizations of the test surmise relations (Section 5) and test knowledge structures (Section 6) have been proposed. Unlike at the level of items (cf. Birkhoff's Theorem 8), at the level of tests, the test surmise relations and test knowledge structures may not necessarily be derived from each other, let alone be set in a one-to-one correspondence. However, (a) each can be characterized by the underlying surmise relation and knowledge structure, (b) the test surmise relations can (not necessarily uniquely) be characterized by the test knowledge structures, and (c) the test knowledge structures can at least 'partly' (under some sufficient condition) be characterized by the test surmise relations.

7.2 Further extensions and modifications

The present approach is purely deterministic. This paper has solely considered combinatorial properties. In future research, probabilistic extensions of the test surmise relations and test knowledge structures and corresponding statistical inference methodologies could be investigated. Such an endeavor constitutes a difficult task, but is indispensable for applications of these models to real data, and may even help to provide feasible new statistical inference methodologies for the common surmise relation and knowledge structure models (at the level of items) in KST. (The latter models are special cases of the presented models.) This, in fact, is an important recent research topic in KST; for instance, see Stefanutti (2006) and Ünlü (2006, 2007) who introduce *item response theory* (e.g., Boomsma, Van Duijn, Snijders, 2001; Fischer & Molenaar, 1995; Van der Linden & Hambleton, 1997) modeling techniques in KST.

The present framework based on tests covers the basic KST models as special instances. Important directions for further research could consider translating even more advanced KST topics into this generalized framework. For instance, the *surmise system* model is a generalization of the surmise relation model (e.g., Doignon & Falmagne, 1999). Unlike the surmise relation model which postulates exactly one mastery strategy for each problem, the surmise system model more realistically allows for different strategies to master a problem. In this spirit, the concept of a test surmise relation might be generalized to the concept of a *test surmise system*.

The introduction of *skills* into the theory of knowledge spaces (Doignon, 1994; Düntsch & Gediga, 1995; Korossy, 1997) represents another interesting topic. A *competence–performance* approach distinguishes between *performance* as the 'empirically observable' solution behavior on a set of selected problems in a particular domain of knowledge, and *competence* (*skills*, *ability*) as the unobservable theoretical entities explaining the solution behavior (performance). An approach based on skills allows for domain-specific qualitative theories to be utilized for performance and competence modeling (qualitative derivation of surmise relation and knowledge structure models). One could try to extend the theory of test knowledge spaces to cover a competence–performance approach, for instance, by introducing 'tests' of skills, as a partition of a set of skills, and linking the 'competence tests' with the 'performance tests' in such a manner that some 'test interpretation and test representation functions' evolve (cf. Korossy, 1997). Such a framework should ideally have as special instances the skills based models proposed in the afore mentioned references.

An interesting equivalent reformulation of surmise relations and knowledge structures and test surmise relations and test knowledge structures is by *Boolean matrices* (Brandt et al., 2003; cf. also Kim & Roush, 1984). The Boolean matrix representation of a surmise relation S on a set Q of items is by a $|Q| \times |Q|$ relational matrix, with entries 1 or 0 depending on whether a pair of items belongs or does not belong to S, respectively. Partitioning the rows and columns of this matrix into tests establishes Boolean matrix representations for the test surmise relations. A knowledge structure $\mathcal{K}$ on Q can be represented by a $|\mathcal{K}| \times |Q|$ Boolean matrix, with entries 1 or 0 depending on whether a knowledge state contains or does not contain an item, respectively. Partitioning the columns of this matrix into tests establishes Boolean matrix representations for the test knowledge structures. The characterizations of the test surmise relations and test knowledge structures presented in set-theoretic and order-theoretic formulations in this paper, could be translated into and investigated within the equivalent formulation by Boolean matrices. This may not only provide other characterizations (in particular, characterizations of the test knowledge structures via the test surmise relations), but also new insights into the studies of (test) knowledge spaces and Boolean matrices. The reformulation of KST and any of its generalizations (e.g., based on skills and/or tests) by (Boolean) matrices, per se constitutes an interesting direction for future research.

7.3 Concluding resume

Though the present approach has the limitation that it does not consider any data analytical, let alone statistical inference method for the derivation and application of test surmise relations and test knowledge structures to simulated or empirical data, we hope to have achieved, however, a number of important characterizations of those models that can definitely help in developing such methods in subsequent work. Having, for instance, data analytically derived one model type (e.g., a test knowledge structure), a characterization formula can be used to derive another model type (e.g., a test surmise relation). With the formulation based on tests, a straightforward interesting generalization of KST is obtained, within which, again by straightforward arguments, useful characterizations of the core models can be derived. That mathematical 'simplicity' comes with a broad range of possible applications of this formulation. The applicability of the tests based models is not only restricted to curriculum development and assessment and training. Interpretations and applications may also be possible in structuring hyper-texts and the organization of companies, and in principle, any field utilizing some types of prerequisite relationships among ‘items’ (e.g., medical diagnosis, failure analysis for a complex system such as a nuclear power plant, and pattern recognition). Discussing such superficially quite different fields when applying the present approach to empirical data in future work, could be a valuable contribution not only in education and psychology but also in just those fields.

Acknowledgments

This work was mostly supported by the Austrian Science Fund (FWF) project grant P12726-SOZ to Dietrich Albert and Wilhelm Schappacher.

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Figure captions

Figure 1. The mastery of problem q implies that of problem r.

Figure 2. Test B is in NTSR–relation to test A ($B \dot{\mathcal{S}} A$).

Figure 3. Reflexivity of NTSRs: Test A is in NTSR–relation to itself ($A \dot{\mathcal{S}} A$).

Figure 4. Transitivity of NTSRs: Test C is in NTSR–relation to test B and test B is in NTSR–relation to test A, but test C is not in NTSR–relation to test A ($C \dot{\mathcal{S}} B$ and $B \dot{\mathcal{S}} A$, but $C \not{\dot{\mathcal{S}}} A$).

Figure 5. Antisymmetry of NTSRs: Test B is in NTSR–relation to test A and test A is in NTSR–relation to test B, but test A is not equal to test B ($B \dot{\mathcal{S}} A$ and $A \dot{\mathcal{S}} B$, but $A \neq B$). In this figure, read $a_1 := a$, $a_2 := b$, $b_1 := c$, and $b_2 := d$.

Figure 6. Test B is in LTSR–relation to test A and test D is in LTSR–relation to test C ($B \dot{\mathcal{S}}_l A$ and $D \dot{\mathcal{S}}_l C$).

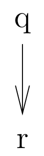
Figure 7. Transitivity of LTSRs: Test C is in LTSR–relation to test B and test B is in LTSR–relation to test A, and this implies that test C is in LTSR–relation to test A ($C \dot{\mathcal{S}}_l B$ and $B \dot{\mathcal{S}}_l A$ implies $C \dot{\mathcal{S}}_l A$).

Figure 8. Test B is in RTSR–relation to test A and test D is in RTSR–relation to test C ($B \dot{\mathcal{S}}_r A$ and $D \dot{\mathcal{S}}_r C$).

Figure 9. Transitivity of RTSRs: Test C is in RTSR–relation to test B and test B is in RTSR–relation to test A, and this implies that test C is in RTSR–relation to test A ($C \dot{\mathcal{S}}_r B$ and $B \dot{\mathcal{S}}_r A$ implies $C \dot{\mathcal{S}}_r A$).

Figure 10. Test B is in TTSR–relation to test A and test D is in TTSR–relation to test C ($B \, \dot{\mathcal{S}}_t \, A$ and $D \, \dot{\mathcal{S}}_t \, C$).

Figures



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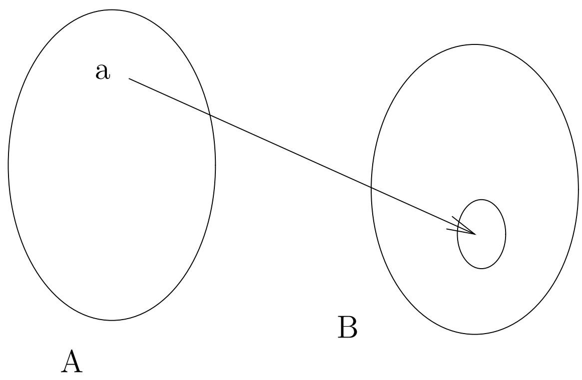

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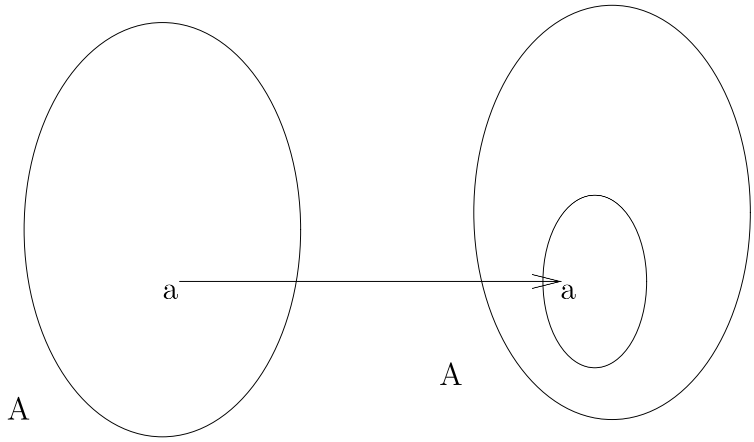

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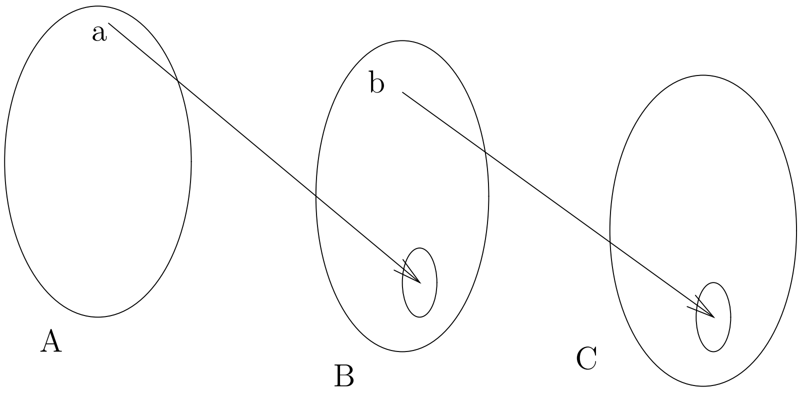

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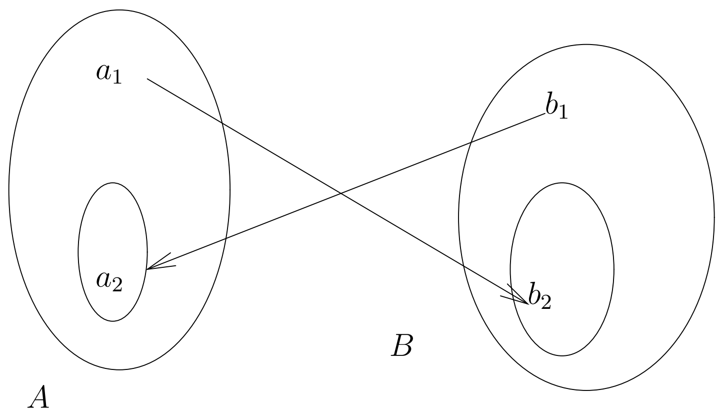

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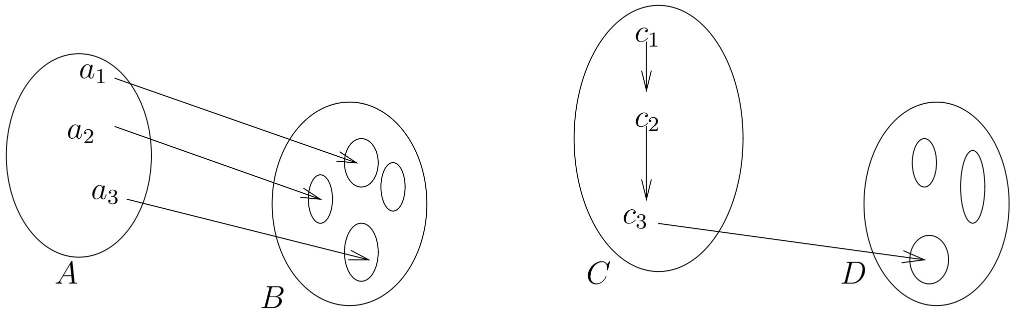


Fig. 6.

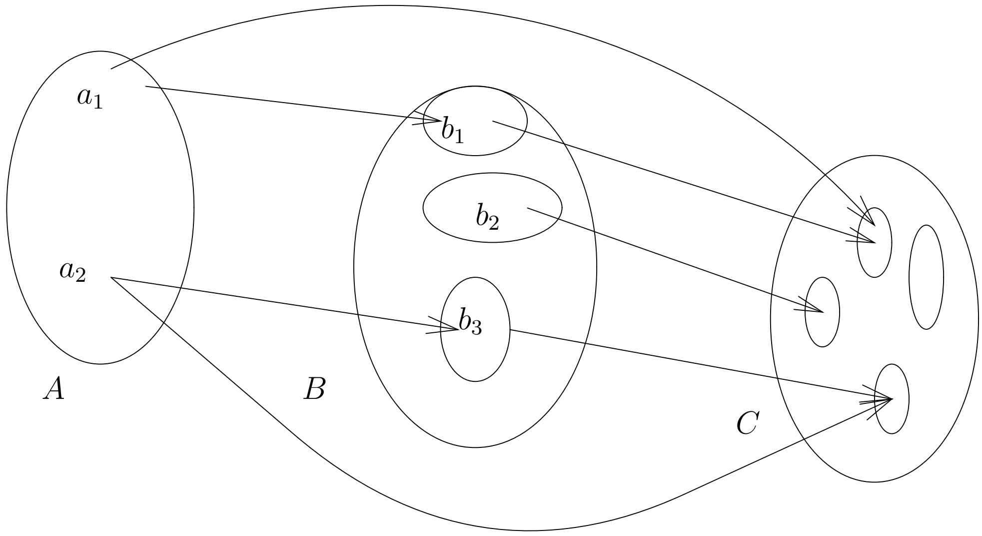
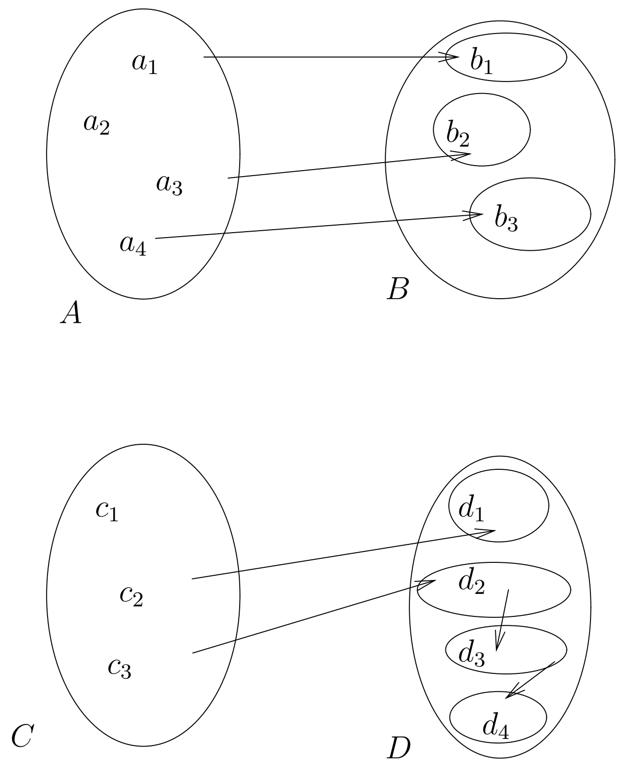


Fig. 7.



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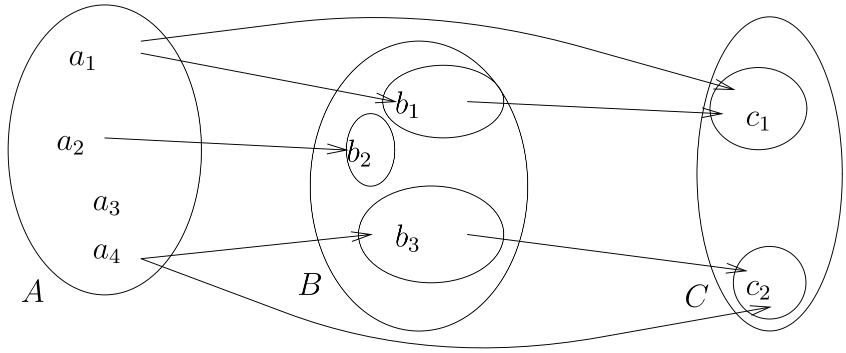

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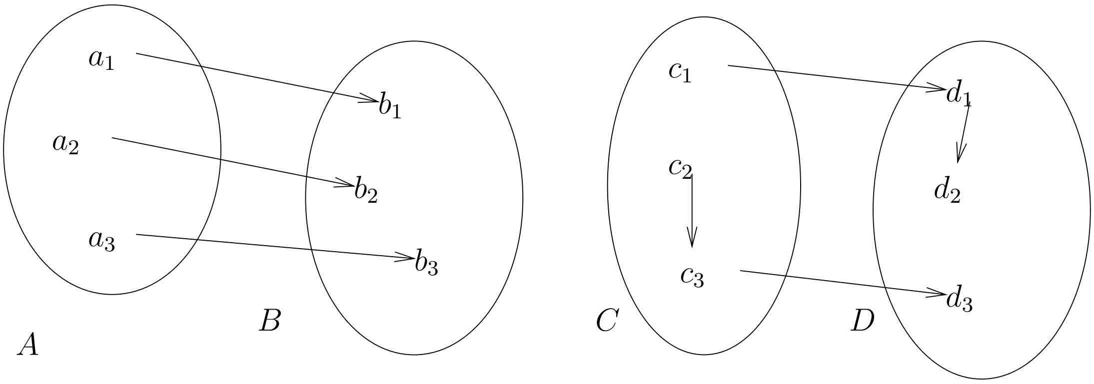

Fig. 10.

Tables

Table 1

Natural test knowledge states

$K \in \mathcal{K}$	$A \cap K$	$B \cap K$	$\dot{K} \in \dot{\mathcal{K}}$
$\emptyset$	$\emptyset$	$\emptyset$	$(\emptyset, \emptyset)$
$\{a_2\}$	$\{a_2\}$	$\emptyset$	$(\{a_2\}, \emptyset)$
$\{b_2\}$	$\emptyset$	$\{b_2\}$	$(\emptyset, \{b_2\})$
$\{a_2, b_2\}$	$\{a_2\}$	$\{b_2\}$	$(\{a_2\}, \{b_2\})$
B	$\emptyset$	B	$(\emptyset, B)$
$\{a_2, b_1, b_2\}$	$\{a_2\}$	B	$(\{a_2\}, B)$
$\{a_1, a_2, b_2\}$	A	$\{b_2\}$	$(A, \{b_2\})$
Q	A	B	(A, B)

Table 2

l-type test knowledge states

$K \in \mathcal{K}$	$A \cap K$	$B \cap K$	$\dot{K}_l \in \dot{\mathcal{K}}_l$
$\emptyset$	$\emptyset$	$\emptyset$	$(0,0)$
$\{a_2\}$	$\{a_2\}$	$\emptyset$	$(1,0)$
$\{b_2\}$	$\emptyset$	$\{b_2\}$	$(0,1)$
$\{a_2, b_2\}$	$\{a_2\}$	$\{b_2\}$	$(1,1)$
B	$\emptyset$	B	$(0,1)$
$\{a_2, b_1, b_2\}$	$\{a_2\}$	B	$(1,1)$
$\{a_1, a_2, b_2\}$	A	$\{b_2\}$	$(1,1)$
Q	A	B	$(1,1)$

Table 3

r-type test knowledge states

$K \in \mathcal{K}$	$A \cap K$	$B \cap K$	$\dot{K}_r \in \dot{\mathcal{K}}_r$
$\emptyset$	$\emptyset$	$\emptyset$	$(0,0)$
$\{a_2\}$	$\{a_2\}$	$\emptyset$	$(0,0)$
$\{b_2\}$	$\emptyset$	$\{b_2\}$	$(0,0)$
$\{a_2, b_2\}$	$\{a_2\}$	$\{b_2\}$	$(0,0)$
B	$\emptyset$	B	$(0,1)$
$\{a_2, b_1, b_2\}$	$\{a_2\}$	B	$(0,1)$
$\{a_1, a_2, b_2\}$	A	$\{b_2\}$	$(1,0)$
Q	A	B	$(1,1)$

Table 4

c-type test knowledge states

$K \in \mathcal{K}$	$A \cap K$	$B \cap K$	$\dot{K}_c \in \dot{\mathcal{K}}_c$
$\emptyset$	$\emptyset$	$\emptyset$	$(0,0)$
$\{a_2\}$	$\{a_2\}$	$\emptyset$	$(1,0)$
$\{b_2\}$	$\emptyset$	$\{b_2\}$	$(0,1)$
$\{a_2, b_2\}$	$\{a_2\}$	$\{b_2\}$	$(1,1)$
B	$\emptyset$	B	$(0,2)$
$\{a_2, b_1, b_2\}$	$\{a_2\}$	B	$(1,2)$
$\{a_1, a_2, b_2\}$	A	$\{b_2\}$	$(2,1)$
Q	A	B	$(2,2)$

Table 5

$B\ \dot{\mathcal{S}}\ A$

$\dot{K} \in \dot{\mathcal{K}}$	$B_K = \emptyset$	$A_K \neq A$
$(\emptyset, \emptyset)$	Yes	**Yes**
$(\{a_2\}, \emptyset)$	Yes	**Yes**

Table 6

$A \dot{\not\mathcal{S}} B$

$\dot{K} \in \dot{\mathcal{K}}$	$A_K = \emptyset$	$B_K \neq B$
$(\emptyset, \emptyset)$	Yes	**Yes**
$(\emptyset, \{b_2\})$	Yes	**Yes**
$(\emptyset, B)$	Yes	**No**

Table 7

$B \; \dot{\$}_l \; A$

$\dot{K}_l \in \dot{\mathcal{K}}_l$	$a_B(K;l) = 0$	$a_A(K;l) = 0$
$(0,0)$	Yes	**Yes**
$(1,0)$	Yes	**No**

Table 8

$A \,\dot{\not{S}}_l\, B$

$\dot{K}_l \in \dot{\mathcal{K}}_l$	$a_A(K;l) = 0$	$a_B(K;l) = 0$
$(0, 0)$	Yes	**Yes**
$(0, 1)$	Yes	**No**

Table 9

$B \, \dot{\mathcal{S}}_r \, A$

$\dot{K}_r \in \dot{\mathcal{K}}_r$	$a_A(K;r) = 1$	$a_B(K;r) = 1$
$(1,0)$	Yes	**No**
$(1,1)$	Yes	**Yes**

Table 10

A $\dot{\mathcal{S}}_r$ *B*

$\dot{K}_r \in \dot{\mathcal{K}}_r$	$a_B(K;r) = 1$	$a_A(K;r) = 1$
$(0, 1)$	Yes	**No**
$(1, 1)$	Yes	**Yes**