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Component-based Knowledge Spaces in Problem Solving and Inductive Reasoning

Dietrich Albert
Karl-Franzens-Universität Graz

Theo Held
Martin-Luther-Universität Halle-Wittenberg

Two principles for component-based problem construction and ordering are presented and applied to different knowledge domains. The first principle is called "set inclusion." It is used for constructing and ordering a set of chess problems. The second principle is "componentwise ordering of product sets." We apply it in the domain of inductive reasoning (i. e. number-series completion). While set inclusion did not prove to be adequate for an application in the considered domain, the componentwise ordering of product sets led to remarkably good results in two empirical investigations.

INTRODUCTION

Procedures that are to test a subject's knowledge concerning a specific domain obviously require—in addition to other prerequisites—a set of problems. The answers to these problems may serve as a basis for a hypothesis about the subject's actual knowledge. A teacher might assume that a student possesses all of the knowledge necessary to solve the problems. There are at least two different methods of questioning:

- All available problems are presented and the set of problems which have been solved correctly is assumed to represent the student's knowledge concerning the investigated domain. This method seems to be rather uneconomical, particularly if the set of problems is quite large.
- The problems that are presented are selected adaptively from a problem set. If a teacher presents a problem which is solved correctly by a student, the next problem will probably be more difficult because the teacher will suppose that the student is capable of solving all easier problems.

Certainly, the second method of knowledge assessment requires an a-priori hypothesis about a *structure* on the problem set. Such a hypothesis may, for example, be: "If a student succeeds in multiplying two fractions, she or he will also be able to multiply two natural numbers." The manner in which a teacher will conduct an assessment procedure depends largely on his or her own experience and knowledge. This experience and knowledge is implicitly used for structuring a knowledge domain. We investigate these hypotheses of a domain's structure in a formal way. First, we take a look at various types of relations that may be defined on a set of problems. This overview serves as a prerequisite for a short introduction to the theory of knowledge spaces put forward by Doignon and Falmagne (1985). Then we will focus on the question of how a relation on a set of problems can be established by using principles for systematical problem construction and discuss two empirical examples.

First, we give some examples of relations on sets of problems. For this purpose, we must introduce a few basic concepts of ordering theory, for example, how can statements like "problem x is more difficult than problem y" or "problem x is at least as difficult as problem y" be denoted?

Most of our examples will involve special cases of quasi-orders (reflexive and transitive). These are partial orders (reflexive, transitive, and antisymmetric), weak orders (transitive and connected), linear orders (connected, antisymmetric, and transitive), and antichains (reflexive, transitive, symmetric, and antisymmetric). In brackets, the defining properties are given from which other properties can be logically derived (e. g., reflexivity of linear and weak orders follows from connectedness). Sometimes irreflexive orders are also used. For a general introduction to ordering theory we recommend of Davey and Priestley (1990).

EXAMPLE 2.1 On a set $Q = \{w, x, y, z\}$ of problems a linear order $\{(w, w),$ $(x, x), (y, y), (z, z), (x, w)\ (y, x), (z, y), (z, x), (z, w), (y, w)\}$ is defined. First, let us look at the Hasse diagram in Fig. 1(a). We can see that this order is a special case of a quasi-order because every problem is comparable to all other problems. Problem w, for example, is supposed to be more difficult than problems x, y, and z. This type of problem ordering is known in psychology as a *Guttman scale* (Guttman, 1947, 1950). □

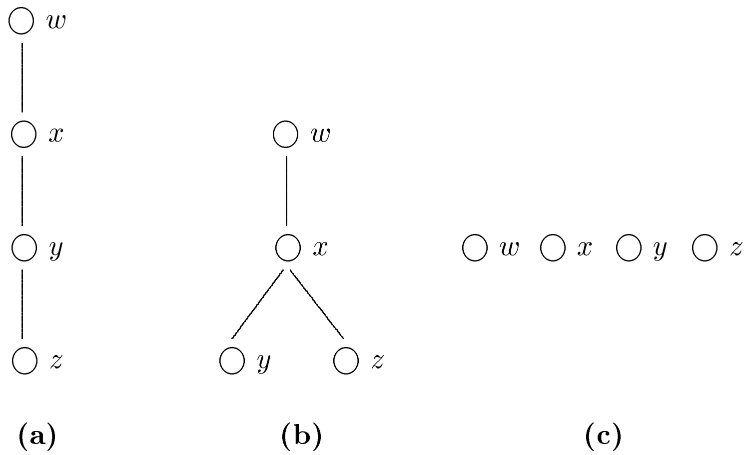

FIG. 1. Hasse diagrams for Examples 2.1, 2.2 and 2.3.

EXAMPLE 2.2 $Q = \{w, x, y, z\}$ is a set of problems with a quasi-order $\{(w,w), (x,x), (y,y), (z,z), (x,w), (y,x), (z,x), (y,w), (z,w)\}$ defined on this set. This order is shown in Fig. 1(b). For our set of questions, this means that problem x should be more difficult than problem y and problem z. It is assumed that problems y and z cannot be compared. □

EXAMPLE 2.3 On a set $Q = \{w, x, y, z\}$ of problems an antichain order $\{(w,w), (x,x), (y,y), (z,z)\}$ is defined. The Hasse diagram in Fig. 1(c) shows that the problems of Q are not connected. Nevertheless, this relation is also a special case of a quasi-order. The postulation of an antichain order may be adequate for sets of heterogeneous and completely incomparable problems. However, it is clear that such a set cannot be used for an economical adaptive questioning procedure because from a subject's answers, no conclusions on other answers can be drawn. □

An important topic in problem ordering is the interpretation of the binary relation that is defined on the problem set. Up until now, we have mentioned only a few unspecified differences in difficulty. In addition to further theoretical considerations, an interpretation will be introduced in the following section.

PROBLEM STRUCTURES AND KNOWLEDGE SPACES

The theory of knowledge spaces (Doignon & Falmagne, 1985, 1998) shows how the structure of problems can be represented in a formal way (for an introduction, see Falmagne, Koppen, Johannesen, Villano & Doignon, 1990).

Let $Q = \{w, x, y, z\}$ be a set of problems that is used for an examination. For some of these problems, a statement such as *"if a student is able to solve a specific problem in Q, he or she will also be able to solve other problems belonging to Q"* may be plausible. This can be formalized in terms of a binary relation $\preceq$. The expression $(y, x) \in \preceq$, which is abbreviated by $y \preceq x$ is interpreted as follows: Given a correct response to problem x, we *surmise* a correct answer to problem y. The relation $\preceq \subseteq Q \times Q$ is called *surmise relation*. It is assumed that the surmise relation is a quasi-order on Q.

A surmise relation can be depicted as a Hasse diagram. The diagrams shown in Fig. 1 can be interpreted as hypothetical orders on the problem set Q. According to the order shown in Fig. 1(b), we assume that each of the students capable of solving problem x will also be able to solve problem y *and* problem z. Based on this assumption, we can collect all subsets of Q that agree with the surmise relation. These subsets are called *knowledge states*.

DEFINITION 2.1 (see Falmagne et al., 1990) Let Q be a set of problems. $K \subseteq Q$ is a *state* $\Leftrightarrow$ $(\forall q, t \in Q, q \preceq t \wedge t \in K \Rightarrow q \in K)$. □

The family of all possible states with respect to a set of problems is a *knowledge structure*. For Example 2.2, we obtain the structure $\mathcal{F}$:

$$\mathcal{F} = \{\emptyset, \{y\}, \{z\}, \{y, z\}, \{x, y, z\}, \{w, x, y, z\}\}.$$

This knowledge structure contains all subsets of Q that are expected to occur as results of diagnostic procedures. The purpose of such procedures is to assign subjects to one of these states without presenting all problems in Q (see Falmagne et al., 1990). $\mathcal{F}$ is closed under union and intersection. A knowledge structure with these properties is called a *quasi-ordinal knowledge space*. A one-to-one correspondence between transitive and reflexive orders and families of knowledge states that are closed under union and intersection is established by a theorem by Birkhoff (1937). The restriction of closure under union and intersection is somewhat unrealistic for many knowledge domains. Therefore, Doignon and Falmagne introduced, as a generalization of quasi-ordinal knowledge spaces, the concept of *knowledge spaces*. Knowledge spaces are families of states that are closed under union, but do not have to be closed under intersection. Hence, every quasi-ordinal knowledge space is also a knowledge space. Doignon and Falmagne showed that there is a one-to-one correspondence between knowledge spaces and the so-called *surmise systems*. This will not be discussed in detail here. Our further considerations will deal solely with quasi-ordinal knowledge spaces.

Quasi-ordinal knowledge spaces can also be derived from "special cases" of quasi-ordered problem sets such as sets with an antichain order or linearly ordered sets.

“COMPONENT-BASED” ESTABLISHMENT OF SURMISE RELATIONS

We will expand our considerations about problems with a topic we call *problem component* or simply *component*. One way to facilitate problem comparison is by *systematical problem construction*. Construction principles are applied on well-defined sets of problem components. Furthermore, by means of their associated component structures, we can both provide a precise description of problems and the class of possible problem variations. Certainly, components have to be equipped with properties that are prerequisites for a successful combination.

Before we introduce two construction principles, a short sketch of the concept that we call a problem component should be drawn. As an example, let us imagine we are asked to solve an algebraic problem, for example, the multiplication of two fractions. Although this is a simple task, we will not be able to give the solution if we do not know some basics of algebra. Some of these basics may be “multiplication of natural numbers”, “division of natural numbers”, and “rules for the multiplication of fractions.”

These items can be seen as *cognitive demands* on a subject confronted with the problem. If the subject does not have the knowledge at his or her disposal which is “demanded” or if the subject is not able to apply this knowledge, it is supposed that the answer to the problem will be incorrect—assuming the guessing probability is equal to zero.

The principle of set inclusion

We now take into account the representation of problems as sets of components. The following examples give a first idea of how problems can be constructed from components.

EXAMPLE 2.4 Let $C = \{a, b, c\}$ be a set of problem components. We assume that an antichain order is defined on C. Let us identify problems with subsets of C. With respect to the antichain order defined on the components, we assume that no dependencies between components exist. Hence, every subset of C can be identified with a potential problem and thus, with an element of a problem set Q (in this case subsets of C denote problems!):

$$Q = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

The combination of the components a, b, and c has led to seven problems. The “empty problem” $\emptyset$ is left out because it cannot be shown. We assume that a problem is more difficult than another problem if it is characterized by all components of the other problem and by at least one more component. According to this assumption, we can state a hypothetical order as shown in Fig. 2(a). The next

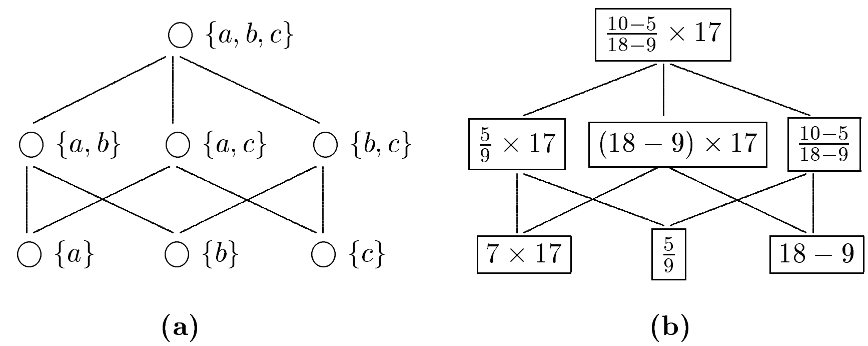

FIG. 2. Problem structure for the questions in Q and an example concerning calculation problems.

step is the application of the construction principle to concrete problem components and the ordering principle on concrete problems. For example, we define for a, b, c: $a \mathrel{\widehat{=}}$ multiplication of numbers, $b \mathrel{\widehat{=}}$ division of numbers, $c \mathrel{\widehat{=}}$ subtraction of numbers. Hence, $\{a, b, c\}$ is a problem that contains multiplication, division and subtraction, for example

$$\frac{10-5}{18-9} \times 17.$$

The hypothetical order for a set of problems is shown in Fig. 2(b). □

We have to stress that a surmise relation was established by constructing and ordering the problems in this way. Unconnectedness, however, is not a necessary property of the component set. As we will show, this method is also applicable to linearly ordered and quasi-ordered component sets. Examples 2.5 and 2.6 give an idea of this method.

EXAMPLE 2.5 We have a set $C = \{a, b, c\}$ of linearly ordered problem components. Fig. 3 shows, on the left, a possible Hasse diagram for the component structure and, on the right, the structure of the resulting problems (components are marked by triangles). Taking into account that the elements of C are linearly ordered, only three problems can be constructed. This linear order may, for example, be induced by constraints on combining the components. Such constraints can exist for sets of non-independent components. In our example, a may be a problem component that also contains b and c in some way. Therefore, if one part of a problem is associated with a, then b and c are automatically involved. To illustrate this we assume that component a corresponds to the addition of natural numbers within the hundreds, for example $619 + 347$. Furthermore we assume that b corresponds to the addition of the numbers between one and ten. We see that b is also necessarily an element of a. Thus, b is an element of problems containing a. □

EXAMPLE 2.6 We assume that a quasi-order is defined on a set $C = \{a, b, c\}$ of problem components. Fig. 4 shows one of the possible Hasse diagrams (left) and the corresponding problem structure (right). From this quasi-ordered problem set, that corresponds to the order shown in Example 2.2, single-component problems consisting either of b or c can be constructed. We may suppose that b and c are thematically independent, but are both involved in a in some way. □

After this brief introduction to one possible method of constructing and ordering problems by means of problem components, we state these principles formally:

DEFINITION 2.2 Let C be a set of components and $\preceq$ a quasi-order on C. The *component space* $\mathcal{F}_C$ is the family of all subsets T of C for which

$$x \in T, y \preceq x \Rightarrow y \in T$$

holds. □

Given a component space $\mathcal{F}_C$, according to each element T of $\mathcal{F}_C$ a problem q_T is formulated. A *surmise relation* R on the problem set $Q = \{q_T \mid T \in \mathcal{F}_C\}$ is defined by the following condition:

$$q_T \; R \; q_{T'} :\Leftrightarrow T \subseteq T'.$$

This means that the problems are identified with the elements of $\mathcal{F}_C$, while the relation R is identified with $\subseteq$.

It is easy to verify that R is a transitive relation: Let M, M', M'' be sets with $M \subseteq M'$ and $M' \subseteq M''$. M' which contains M is a subset of M'', thus $M \subseteq M''$, which means that '$\subseteq$' is transitive.

This ordering principle of set inclusion is based on the plausible assumption that a subject succeeding in the solution of a given problem will also be able to solve all the subproblems of this problem. We have to note here that a reversed

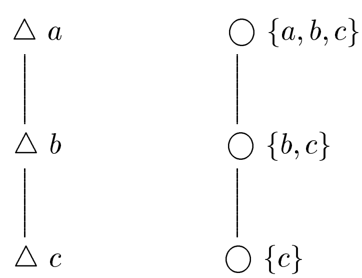

FIG. 3. Component structure and problem structure for Example 2.5.

statement such as "if someone is able to solve the subproblems, she or he will also be able to solve the superset problem" is not expected to hold true. The combination of problem components may lead to some additional difficulties that might not appear within the single components.

As an ordering method, set inclusion can be applied to very different theoretical approaches in the field of knowledge assessment. For examples, we refer to the investigations of Korossy (1993, 1997).

The principle of componentwise ordering of product sets

Up until now, we have focused only on single sets of components that were characterized by an order that was defined on the component set. In this section, we turn our attention to the construction of problems that consist of components with variable attributes. Here, every problem is equipped with the same number of components. New problems are constructed by varying the attributes of the components. The order of the problems will be derived from relations that are defined on the set of attributes. The following example gives an idea of this method.

EXAMPLE 2.7 Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2\}$ be problem components; a_1, a_2, a_3 and b_1, b_2 are the attributes of these components. On both sets A and B, a linear order is defined (see the left side of Fig. 5; attributes are marked by black triangles).

Suppose we want to construct simple algebra problems. One component may be the set of numbers that is used within a calculation, the other component is characterized by the operations that are to be applied on the set of numbers. We define: $a_1 \mathrel{\widehat{=}}$ use of real numbers, $a_2 \mathrel{\widehat{=}}$ use of integers, $a_3 \mathrel{\widehat{=}}$ use of natural numbers, $b_1 \mathrel{\widehat{=}}$ calculation of powers, $b_2 \mathrel{\widehat{=}}$ addition.

Both operations of B can be applied on the sets of numbers of A. Therefore, we can construct problems that contain one property of A and one property of B. The problem $(-5)^2$, for instance, corresponds to the combination of a_2 and b_1.

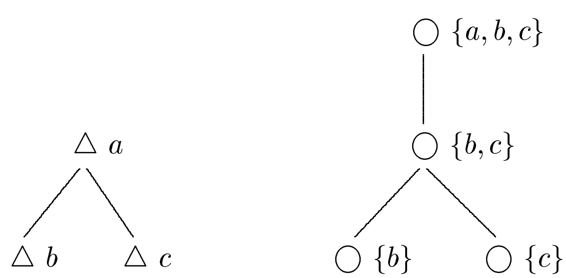

FIG. 4. Component structure and problem structure for Example 2.6.

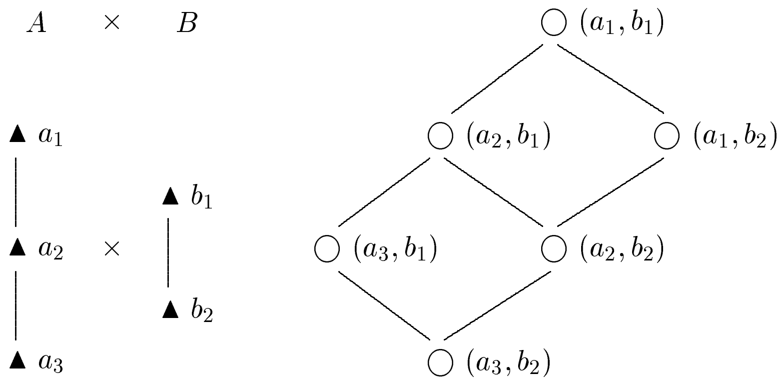

FIG. 5. Orders of attributes and problem structure based on componentwise ordering for Example 2.7.

From A and B we can construct a set $\mathcal{F}_p$ of six problems:

$$\mathcal{F}_p = \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2), (a_3, b_1), (a_3, b_2)\}.$$

We see that all problems of $\mathcal{F}_p$ consist of two components that are represented by their attributes $a_1, \ldots, a_i, \ldots, a_n$ and $b_1, \ldots, b_j, \ldots, b_m$. The problem structure for this example is shown in Fig. 5 (on the right-hand side). □

Let us now consider the principles by which these problems were constructed and ordered in detail. Components are sets of attributes. It is supposed that the attributes in a component cannot be combined with one another. In our example we have two components whose attributes are combined. This combination has been established by forming the *Cartesian product* of A and B. Looking at Example 2.7, we can easily check that $\mathcal{F}_p$ is a set that contains the product of A and B.

In order to establish a surmise relation as shown in Fig. 5, it is necessary to compare the generated problems in pairs with respect to the attributes of the components. Formally, the ordering rule we applied was:

> Let $C_1, \ldots, C_n$ be component sets on which partial orders $R_1, \ldots,$ R_n are defined. On the Cartesian product $C_1 \times \ldots \times C_n$ an order $\preceq$ is imposed by defining
>
> $$(x_1, \ldots, x_n) \preceq (y_1, \ldots, y_n) \Longleftrightarrow (\forall i)\; x_i \; R_i \; y_i.$$

Expressed in words: We surmise that a problem q_1 is at least as difficult to solve as a problem q_2, if all attributes of q_1 are at least as difficult as the corresponding attributes in q_2 with respect to the relations R_i defined on the attribute sets. This principle is known as "coordinatewise order", for a description see Davey and

Priestley (1990, p. 18). According to Birkhoff (1973), $\preceq$ is a partial order (i. e. reflexive, transitive, and antisymmetric). Note that this method is also known from decision theory where the choice heuristic called *dominance rule* corresponds to coordinatewise orders.

Extensions of this ordering method applied to problems of elementary probability calculus are introduced in Held (1992, 1993). There, an approach to the component-based establishment of surmise systems can also be found.

Because the attributes of the components must be compared, it is necessary to define an order on each set of attributes. Example 2.7 showed the case of linearly ordered attributes. Example 2.8 demonstrates the ordering of problems that were constructed from quasi-ordered sets of attributes.

EXAMPLE 2.8 Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ be partial-ordered sets of attributes. Fig. 6 shows the Hasse diagrams for these sets and the corresponding problem structure. A procedure for the graphical construction of such products is given in Davey and Priestley (1990, p. 19). □

Problems that were constructed by product formation can also be ordered *lexicographically*.

EXAMPLE 2.9 As in Examples 2.7 and 2.8, we have two components $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2\}$. It is assumed that component A is "more important" than component B. Fig. 7 shows the lexicographic order of the product $A \times B$. How was this order established? First, we describe the general principle. The n-tuples that are to be ordered are compared pairwise beginning with the first elements (here: a_i). Because it is assumed that A is the most "important" component, it is also assumed that if these elements are not identical, the n-tuple that contains the subordinate element with respect to the order on A is subordinate to the other n-tuple. In Fig. 7 we see that this is the case for all tuples (a_1, b_i) and (a_2, b_j). If the first elements are identical, the second pair of elements will be compared and the n-tuple with the subordinate element is subordinate (see all tuples (a_i, b_1) and (a_i, b_2)). This procedure that is known from dictionaries continues on until two different elements are found or until there are no more elements left to compare. □

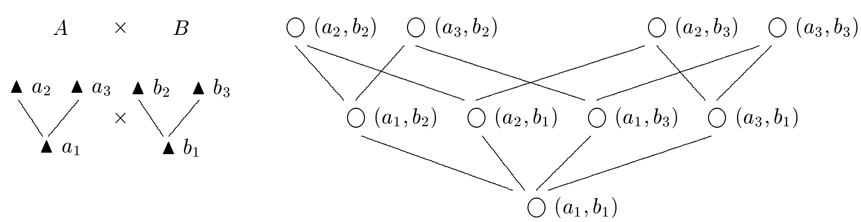

FIG. 6. Attributes and problem structure for Example 2.8.

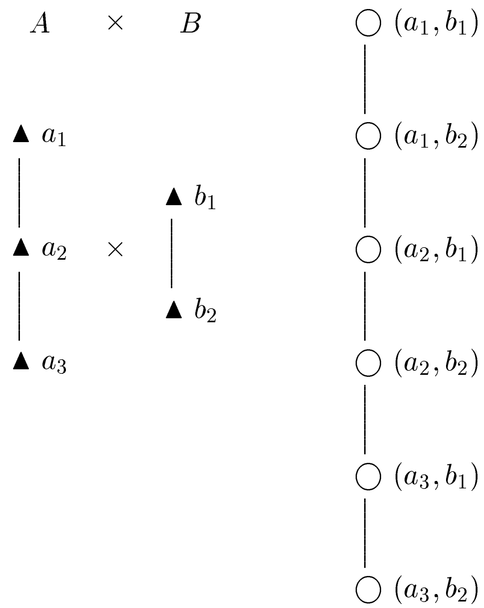

FIG. 7. Orders of attributes and problem structure based on lexicographic ordering.

For establishing a lexicographic order we need sets A_i of attributes and relations P_i that are defined on the sets A_i. $\langle A_i, P_i \rangle$ with $i = 1, \ldots, n$ have to be strict linear orders (i. e. irreflexive, transitive, and weakly connected). A lexicographic order is defined as follows:

DEFINITION 2.3 Let $\langle A_i, P_i \rangle, i = 1, \ldots, n$, be strict linear orders. $\langle A_1 \times A_2 \times \ldots \times A_n, L \rangle$ with $L \subseteq ((A_1 \times A_2 \times \ldots \times A_n) \times (A_1 \times A_2 \times \ldots \times A_n))$ is a *lexicographic order*, that is for

$$(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n) \in A_1 \times A_2 \times \ldots \times A_n$$

holds $(a_1, a_2, \ldots, a_n)\ L\ (b_1, b_2, \ldots, b_n)$ if and only if,

$$\begin{aligned} & a_1 P_1 b_1 \ \vee\ (a_1 = b_1 \ \wedge\ a_2 P_2 b_2) \ \vee\ (a_1 = b_1 \ \wedge\ a_2 = b_2 \ \wedge\ a_3 P_3 b_3) \ \vee\ \ldots \\ & \qquad \ldots \vee\ (a_1 = b_1 \ \wedge\ a_2 = b_2 \ \wedge\ \ldots \ \wedge\ a_{n-1} = b_{n-1} \ \wedge\ a_n P_n b_n) \\ & \qquad\qquad \vee\ (a_1 = b_1 \ \wedge\ a_2 = b_2 \ \wedge\ \ldots \ \wedge\ a_n = b_n). \end{aligned}$$

□

We introduced two important principles (i. e. set inclusion and componentwise ordering of product sets) for the construction of problems from components and for the establishment of a surmise relation on these problems. In the following section, we present two empirical investigations that make use of those principles.

EMPIRICAL EXAMPLES

The empirical examples report on experimental investigations which make use of the principles introduced for problem construction and problem ordering. The first investigation belongs to the area of psychology of problem solving. It deals with the solution of chess problems. In the second experiment, we focus on types of problems related to the field of inductive reasoning: the continuation of number series.

Construction and solution of chess problems

Chess playing is surely one of the most complex and demanding knowledge domains. This complexity makes the domain particularly interesting for cognitive scientists and psychologists. Not only the game of chess itself, but also the construction of chess problems requires a large amount of knowledge and experience. The immense number of possible moves that can be made, even from a very simple constellation, makes the decision, of whether one move is better than another very difficult. Grandmasters are often unable to "proof" what move is the best in a particular situation; therefore they often have to act intuitively.

An important book about the psychology of chess playing was written by De Groot (1965). He attempted to investigate the thought processes of highly trained chess players by means of introspective methods. De Groot also provided a proof scheme for objectively solvable positions, but the proof only works if someone is able to differentiate between "good" and "less good" moves. This differentiation has, for complex positions, to be intuitive.

We can already see that for the construction of chess problems, we should not attempt to focus on such demanding constellations that in addition to requiring highly evolved skills, are also very time consuming. In our example, we use the classical form of "three move problems" that are familiar to every chess player. In Fig. 8, we provide a typical example. The task is to perform the moves to reach a "winning position in three moves." Supposing white starts, the solution is: 1. Be2 h1Q; 2. Bh5+ Qh5:; 3. Ng7+. Experienced chess players can show that for this type of problem there is only one optimal solution. Furthermore, the time needed for handling such a position is expected to be much shorter than for a complex constellation in a real chess game.

As a next step, we have to find a way for the construction of such problems. Before we can apply one of our construction rules in the next section, components have to be introduced. A basic concept in chess playing are "*motives*", which are tactical standard situations. In terms of problem solving, motives can be seen as subgoals of a problem's solution. Fig. 9 shows examples for positions in which the motives "fork", "pin", "guidance", and "deflection" occur.[1] To illustrate we

[1] The example positions are identical to problems of the experimental investigation. Therefore

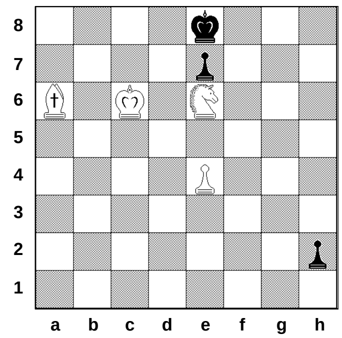

FIG. 8. A typical three move problem.

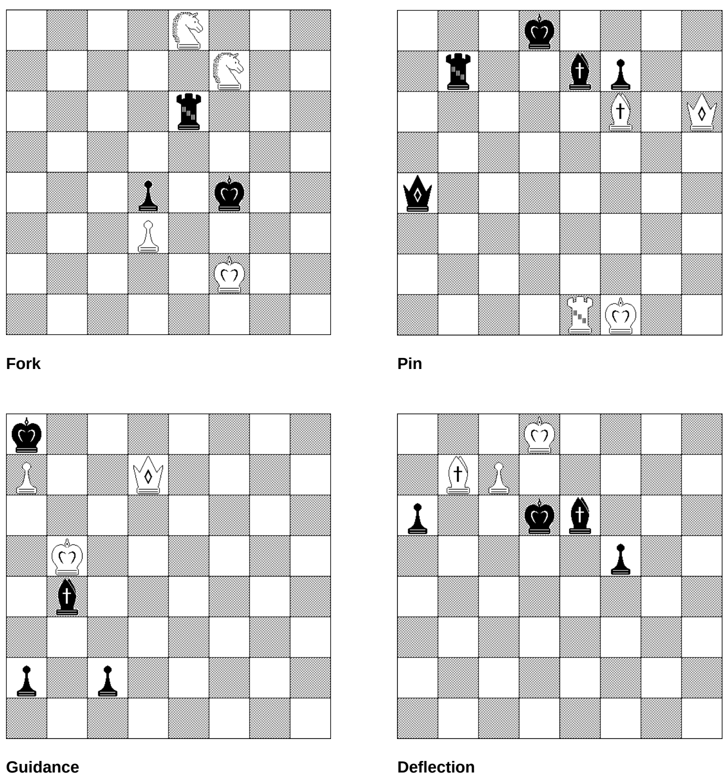

FIG. 9. Positions in which the motives “fork”, “pin”, “guidance”, and “deflection” occur.

give a short description of these special situations:

- *Fork:* One piece simultaneously attacks two opposing pieces of higher value. Solution:[2] 1. Nc7 Rg6/c6; 2. Nd5+ arbitrary[3]; 3. Ne7+/Ne5+. If we take a look at one of the possible final positions (*Black:* Kf5, Rc6, …*White:* Ne7, …), we see that White's Knight attacks both Kf5 and Rc6.
- *Pin:* An opposing piece is prevented from moving. Solution: 1. Qf8+ Qe8; 2. Rd1+ Rd7; 3. Be7:+. We see that the black Bishop cannot move away from e7 because of Bf6+.
- *Guidance:* An opposing piece is forced to a disadvantageous square. Solution: 1. Kb6 Ba5+/c5+; 2. Ka6c6 arbitrary; 3. Qb7/c6 mate; Black's Bishop is forced to a5+/c5+, otherwise 2. Qb7 mate.
- *Deflection:* An opposing piece is forced to leave an important line or square. Solution: 1. Bc8 Bd5; 2. Bf5: Bb7; 3. Be4. Black's Bishop is forced to leave e6, otherwise 2. Be6: ...

Motives can appear in a large variety of combinations and belong to the basic repertory of even only moderately experienced chess players. A complete list of all problems used in our investigation can be found in Table 1.

For the construction of problems, these motives present one possible type of problem components. As a principle of construction, we select a small number of motives and then produce three move problems that contain *combinations* of them. In the following, an investigation that makes use of this idea, is reported.[4]

Problem Construction and Hypothesis

As we have already indicated, the construction of problems and the establishment of the surmise relation are based on the combination of motives. The motives—symbolized by a, b, c, and d—are elements of a single component set C. We assume that an antichain order is defined on C. The principle of set inclusion will be applied.

The component space $\mathcal{F}_C$ (see Definition 2.2) is as follows:

$$\begin{aligned}\mathcal{F}_C = \quad & \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \\ & \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, \{a,b,c,d\}\}.\end{aligned}$$

By means of the ordering principle of set inclusion as introduced above we can infer a surmise relation R on the set Q of the 15 problems that are identified with the elements of $\mathcal{F}_C$. Fig. 10 shows this relation as a Hasse diagram. Expressed in words, the hypothesis for the investigation is:

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these examples may appear to be rather complex.

[2] The "solution" provides the sequence of three moves which a chess expert has considered as optimal for reaching a winning position.

[3] "Arbitrary" means that this move (in this case Black's move) is not relevant to the solution.

[4] The investigation was conducted by B. Hierholz at the University of Heidelberg under direction of the first author.

TABLE 1
Complete List of Chess Problems.

Number	*Type*	*Position*	*Solution*	*Motives*
1	abcd	*White*: Ka7 Qh3 Re5 Nd6 *Black*: Kh8 Qg6 Rg8 Bf7 Ph7	1. Rg5 Qf6 2. Qc3 Qc3: 3. Nf7 mate	deflection, guidance, pin, fork
2	bcd	*White*: Kh2 Bf3 Nh5 Pg3,g7 *Black*: Kh7 Qe6 Ph3	1. g8Q+ Kg8: 2. Bd5 Qd5: 3. Nf6+	guidance, pin, fork
3	abc	*White*: Kg1 Qe2 Re1 Bg6,h2 Pf2 *Black*: Kf8 Qb7 Rg8 Be7,h3 Pg7,f6	1. Qe7:+ Qe7: 2. Bd6 Qd6: 3. Re8 mate	deflection, guidance, pin
4	acd	*White*: Kg1 Qc2 Rf2 Bb1 Nf8 Pb2,c6,g2 *Black*: Kd8 Qg7 Rd6 Be4 Nd3 Pb7,e7	1. cb: Bb7: 2. Qd3: Rd3: 3. Ne6+	deflection, pin, fork
5	abd	*White*: Ka2 Qf4 Be3 Pb2,b3,h3,c7 *Black*: Ka5 Qe7 Nb6 Pa6,b5,c5,b4,h4	1. Qb4:+ cb: 2. Bb6:+ Kb6: 3. c8N+	deflection, guidance, fork
6	bc	*White*: Kf1 Qa6 Re1 Nh3 Pg2,f2,d4 *Black*: Ke8 Qd6 Rh8 Nc6 Pe6,f7,g7	1. d5 Qd5: 2. Qa8+ arbitrary 3. Qc6:+/Qd5:/Qh8:	 guidance, pin
7	ad	*White*: Kd6 Nf5 Pe7 *Black*: Kf7 Ng4 Ph7	1. Nh6+ Nh6: 2. Ke2 arbitrary 3. e8Q	 deflection, fork
8	bd	*White*: Kc6 Ba6 Ne6 Pe4 *Black*: Ke8 Pe7,h2	1. Be2 h1Q 2. Bh5+ Qh5: 3. Ng7+	 guidance, fork
9	ac	*White*: Kh2 Bb6 Pf3,g2 *Black*: Kh4 Rc2 Ph7,h5,g5	1. Bc7 Rg2:+ 2. Kg2: arbitrary 3. Bd8/f2 mate	 deflection, pin
10	cd	*White*: Kf3 Rc6 Ne5 Pg5 *Black*: Kg8 Rd4 Be7 Pf4	1. Rc8+ Kg7 2. Rc7 Kf8 3. Ng6+	 pin, fork
11	ab	*White*: Kh2 Qd1 Re2 Pd7,f2,h4 *Black*: Kg8 Qb5 Rd8 Pa4,g7,h7	1. Re8+ Re8: 2. Qd5+ Qd5: 3. deQ mate	 deflection, guidance
12	d	*White*: Kf2 Ne8,f7 Pd3 *Black*: Kf4 Re6 Pd4	1. Nc7 Rg6/c6 2. Nd5+ arbitrary 3. Ne7+/Ne5+	 fork
13	c	*White*: Kf1 Qh6 Re1 Bf6 *Black*: Kd8 Qa4 Rb7 Be7 Pf7	1. Qf8+ Qe8 2. Rd1+ Rd7 3. Be7:+	 pin
14	b	*White*: Kb5 Qd7 Pa7 *Black*: Ka8 Bb4 Pa2,c2	1. Kb6 Ba5+/c5+ 2. Ka6/c6 arbitrary 3. Qb7/c6 mate	 guidance
15	a	*White*: Kd8 Bb7 Pc7 *Black*: Kd6 Be6 Pf5,a6	1. Bc8 Bd5 2. Bf5: Bb7 3. Be4	 deflection

Problem 8 by Maiselis and Judowitsch (1966); problem 10 by Geisdorf (1984); problem 12 by Chéron (1960); problem 14 by Speckmann (1958); the other problems by B. Hierholz.

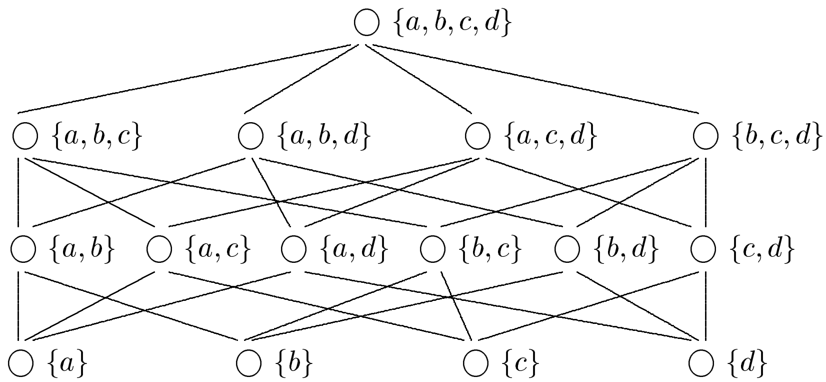

FIG. 10. Hasse diagram for the problems identified with the elements of the component space $\mathcal{F}_C$.

> *If a problem q identified with a component set $T \in \mathcal{F}_C$ is solved by a subject, then all problems q' which are identified with a component set $T' \in \mathcal{F}_C$ with $T' \subseteq T$ will also be solved by this subject.*

The hypothetical structure corresponds to a set of 167 knowledge states. This means that only about 0.5 % of $2^{15} = 32768$ potential solution patterns represent valid states.

Method

For the investigation, four motives were selected and combined as shown in Fig. 10. These motives are "fork", "pin", "deflection" and "guidance." The combinations of these four motives form a set of 15 problems. The complete list of the problems is given in Table 1. The positions of Fig. 9 are examples for the problems with only one motive. Fig. 8 presents a problem with the two motives, "guidance" and "fork." These problems were presented to 13 subjects who were all members of the chess club in Ladenburg, Germany.

First, the subjects were asked to read the instructions for the experimental procedure, then they were permitted to begin working on the problems. Each problem was printed on a single card as a diagram (see Fig. 8 and 9). The subjects had to write down the solution in the usual form used above.

The time needed for the solution was controlled by the subjects themselves with the aid of a chess clock. There was no time limit. The subjects were asked only to answer "as accurately and as quickly as possible." The problems were presented in the order of hypothesized difficulty, so problem $\{a,b,c,d\}$ with four motives was the first to be presented and the one-motive problems $\{a\}$, $\{b\}$, $\{c\}$, and $\{d\}$ were the last to be presented.

Results

The criteria for the goodness of fit that will be used in this paper are (1) the *minimal distances* between each solution pattern and the closest states in the quasi-ordinal knowledge space and (2) the *mean distance* between the set of solution patterns and the states in the quasi-ordinal knowledge space. Let $\mathcal{K}$ be a quasi-ordinal knowledge space, and let $\mathcal{S}$ be the set of all solution patterns in the data; let the elements of $\mathcal{K}$ as well as those of $\mathcal{S}$ be represented in the form of subsets of problems. Further, let $X \in \mathcal{S}$ and $K \in \mathcal{K}$. Then the *distance* between X and K, abbreviated by *dist*(X, K), is defined as the number of elements occurring in the symmetric set difference of X and K, that is,

$$\textit{dist}(X, K) = |(K \setminus X) \cup (X \setminus K)|.$$

The *minimal distance* of X to $\mathcal{K}$, abbreviated by *mdist*$(X, \mathcal{K})$, is defined as the distance of X to the "nearest" knowledge state in $\mathcal{K}$, that is,

$$\textit{mdist}(X, \mathcal{K}) = \min\{\textit{dist}(X, K) | K \in \mathcal{K}\}.$$

Now, the *mean distance* $d(\mathcal{S}, \mathcal{K})$ of $\mathcal{S}$ to $\mathcal{K}$ is given by

$$d(\mathcal{S}, \mathcal{K}) = \sum\{\textit{mdist}(X, \mathcal{K}) | X \in \mathcal{S}\}/N,$$

where N is the number of response patterns in $\mathcal{S}$. Table 2 shows the frequencies of minimal distances and and the mean distance d for the solution patterns of the 13 subjects and the 167 knowledge states of the hypothetical knowledge space.

The hypothesis holds only for three subjects who solved all problems and for one subject who failed only in solving problem $\{a, b, c, d\}$. Two subjects out of 13 each show inconsistencies for only one problem.

Discussion

The results clearly contradict our deterministic hypothesis because the response patterns of only four subjects agree with it. The reasons for the unsatisfactory results may be found both in the theoretical approach and the experimental design. First of all, the difficulty of the chess problems is probably not solely

TABLE 2
Chess Problems: Minimal Distances and Mean Distance Between the Solution Patterns and the Hypothetical Knowledge Space

	Distances							d
	0	1	2	3	4	5	6	
Frequencies	4	2	1	3	2	0	1	2.08

influenced by the type and number of included motives. An investigation by Albert, Schrepp, and Held (1994) show that taking the sequence of motives within problems into consideration can contribute to a more adequate problem structure.

Another problem common to investigations dealing with chess playing is that the work on chess problems requires great concentration over a large period of time. Thus, we suspect that the order of problem presentation (beginning with $\{a, b, c, d\}$) might not have been the best choice. The experimental setting as a group experiment and the lack of any limit on solution times may have caused a decrease in motivation with some of the subjects who required longer solution times.

In the investigation of Albert, Schrepp, and Held (1994) mentioned above, these problems were taken into account. A computerized experimental laboratory setting was used. Further, the uniqueness of motive assignment was optimized. Due to these improvements the results of this investigation are much more conclusive than the ones reported here.

Continuing a series of numbers

Our second empirical example deals with a type of task that is commonly found in diagnostic instruments in psychology. It is typical for inductive reasoning. A series of numbers constructed according to an algebraical rule is to be continued by one or more numbers. Subjects are required to infer the rule from the number series presented and to calculate the missing number with the help of this rule. The following example demonstrates a very simple task:

$$30\ 32\ 36\ 44\ 60\ \ldots ?$$

One possible rule is: $x_n = x_{n-1} + 2^n$. Of course, we can find other formulas that correspond to the example, e. g. $x_n = 3x_{n-1} - 2x_{n-2}$, where x_n is the number, we are trying to find, x_{n-1} is the preceding number (here: 60), and so on. Our example shows that both formulas use preceding elements of the given series for the calculation of x_n. We call the number of immediate predecessors that are used for the solution of the problem the *level of recursion*. The first formula has recursion level “1”, the second is of level “2.” Krause (1985) used this type of recursively connected number series in an investigation of mental processes and rule detection. He attempted to classify the various methods subjects used to solve this type of problem.

Some types of number series problems possess properties that make them suitable for our component-based method of problem construction. The level of recursion is one of them. Generally, we assume that the following cognitive demands are covered by number series problems: (1) the subject has to recognize properties and regularities of the presented sequence (e. g. the level of recursion),

TABLE 3
Number Series: Problem Components

Components	*Attributes*		
$\mathbf{M_1}$	$\mathbf{a_1}$	$\mathbf{a_2}$	$\mathbf{a_3}$
	level of rec.: 3	level of rec.: 2	level of rec.: 1
$\mathbf{M_2}$	$\mathbf{b_1}$	$\mathbf{b_2}$	
	multiplicative factor	multiplicative factor	
	$\mathrm{f} > 1 \wedge \mathrm{f} \in \boldsymbol{N}$	$\mathrm{f} = 1$	
$\mathbf{M_3}$	$\mathbf{c_1}$	$\mathbf{c_2}$	
	additive factor	additive factor	
	$\mathrm{g} > 1 \wedge \mathrm{g} \in \boldsymbol{N}$	$\mathrm{g} = 0$	

(2) a hypothesis concerning the underlying rule has to be established, applied, and tested.

Problem construction and hypothesis

Number series problems are extremely variable, so the question is, what types of components can be combined in which ways. In this investigation, three distinct components M_1, M_2, M_3 were used. Their attributes are shown in Table 3. Concerning attributes b_2 and c_2, we must note that the definition of the factors $f = 1$ and $g = 0$ is included for "technical" reasons: although a recognition of a multiplicative or additive factor is not necessary for a solution of the problems which are characterized by b_2 or c_2, giving such "zero values" is appropriate for a complete problem definition by elements of a Cartesian product. We assume that a linear order is defined on the attributes of each component. The Hasse diagrams of Fig. 11 (left) illustrate this fact. This assumption means that, for example, recursion level 3 makes a problem more difficult than recursion level 2, or the existence of a multiplicative factor that is greater than 1 provides more complication than factor 1 that psychologically corresponds to no demand for detecting this factor.

Now we must define a problem construction rule for these components. In the previous section, we demonstrated how product formation can be applied to sets of components. We apply this rule to M_1, M_2, M_3. The product $M_1 \times M_2 \times M_3$ provides twelve combinations of attributes of the type (a_n, b_n, c_n). We call the set of these combinations *problem set* Q_t.

The next step is the application of the *componentwise ordering rule*. This leads to the structure of the twelve problems, where problem (a_1, b_1, c_1) is assumed to be the most difficult and (a_3, b_2, c_2) the simplest. Table 4 shows the complete problem set constructed for the investigations. On the right side of Fig. 11, we can see the problem structure.

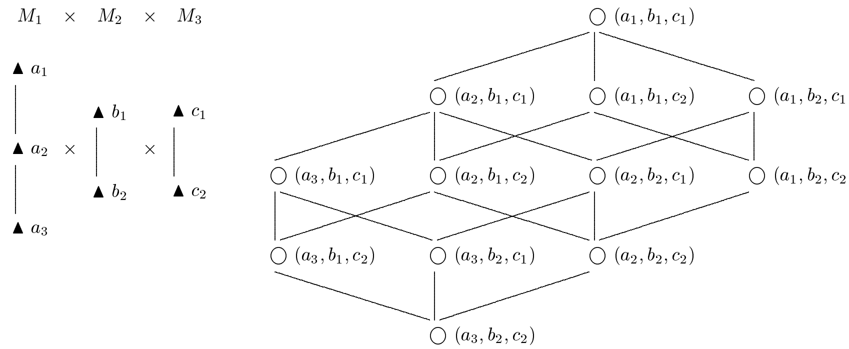

FIG. 11. Number series: Orders of attributes and problems.

TABLE 4
Number Series: Calculation Rules, Problems, and Solutions

Attributes		c_1	c_2
a_1	b_1	$x_n = 2x_{n-3} + x_{n-2} + x_{n-1} + 4$ $1, 5, 9, 20, 43, 85 \Rightarrow 172$	$x_n = 2x_{n-3} + x_{n-2} + x_{n-1}$ $6, 6, 7, 25, 44, 83 \Rightarrow 177$
	b_2	$x_n = x_{n-3} + x_{n-2} + x_{n-1} + 1$ $16, 16, 17, 50, 84, 152 \Rightarrow 287$	$x_n = x_{n-3} + x_{n-2} + x_{n-1}$ $26, 34, 41, 101, 176, 318 \Rightarrow 595$
a_2	b_1	$x_n = x_{n-2} + 2x_{n-1} + 2$ $1, 4, 11, 28, 69, 168 \Rightarrow 407$	$x_n = 2x_{n-2} + x_{n-1}$ $5, 11, 21, 43, 85, 171 \Rightarrow 341$
	b_2	$x_n = x_{n-2} + x_{n-1} + 5$ $12, 17, 34, 56, 95, 156 \Rightarrow 256$	$x_n = x_{n-2} + x_{n-1}$ $25, 34, 59, 93, 152, 245 \Rightarrow 397$
a_3	b_1	$x_n = 2x_{n-1} + 1$ $7, 15, 31, 63, 127, 255 \Rightarrow 511$	$x_n = 2x_{n-1}$ $4, 8, 16, 32, 64, 128 \Rightarrow 256$
	b_2	$x_n = x_{n-1} + 13$ $33, 46, 59, 72, 85, 98 \Rightarrow 111$	$x_n = x_{n-1}$ $113, 113, 113, 113, 113 \Rightarrow 113$ (not presented in Investigation I)

Method

The problems used in the investigations have the following properties: the multiplicative constant is either 2 or 0, the additive constant is always a single- or two-digit element of $\boldsymbol{N}_0$, and the maximal recursion level is 3. To avoid successful guessing of the solution, all solutions are numbers greater than 100. Table 4 shows all calculation rules and the corresponding problems. The problems were used in two investigations. Investigation I took place at the Department of Psychology of the University of Heidelberg, Germany, with 18 subjects. Investigation II was conducted with 30 subjects at the Department of Psychology of the University of Graz, Austria.[5]

In Investigation II, the complete set of 12 problems was presented, where as in Investigation I the trivial problem (a_3, b_2, c_2) was not used assuming that it would possibly have confused the subjects.

The hypothetical structure $\mathcal{K}_1$ for 11 problems (as used in Investigation I) corresponds to 49 knowledge states. This is 2.4 % of $2^{11} = 2048$ possible solution patterns. In the case of 12 problems (Investigation II), we have a hypothetical structure $\mathcal{K}_2$ with 50 knowledge states (i. e. 1.2 % of $2^{12} = 4096$ possible patterns).

In both investigations, the subjects were asked to read instructions that introduced the problem type. The subjects were also told that the only mathematical operations to be used were addition and multiplication with non negative numbers. Then they were asked to solve three simple example problems.

The problems were then presented (printed on cards) in a randomized order. The subjects had to write down the solution on a sheet of paper. If the solution was not given within 7 minutes, the next problem was presented. Subjects who either gave the wrong solution or wanted to "give up" before the 7 minutes had expired were asked to go on thinking about the problem. After the last problem, the subject was asked for the rules he or she used for solving the problems.

Results[6]

None of the subjects in Investigation II failed solving the trivial problem (a_3, b_2, c_2). Therefore, we ignore this problem in our further analysis and consider only knowledge space $\mathcal{K}_1$ for the comparisons between data and hypothetical structure.

In Table 5, the results of the two investigations are presented separately. In addition the results for the whole group of 48 subjects are shown. We see that in Investigation I the solution pattern of 16 out of 18 subjects is identical with a knowledge state in $\mathcal{K}_1$. Only patterns of two subjects have a distance of 1 to a state in $\mathcal{K}_1$. In Investigation II, solution patterns with distances of 2 and 3 also occurred. All in all, 14 different solution patterns with a distance of 0 have been observed.

Discussion

The results of this investigation show that the hypothetical conclusions we drew about the componentwise ordering rule were rather accurat.

[5] Investigation I was conducted by P. Hellriegel, J. Ptucha and M. Wölk. Investigation II was conducted by Birgit Edlinger and Dagmar Spreitzhofer. Both investigations were guided by the first author.

[6] We are grateful to Jochen Musch (University of Bonn), who conducted a first analysis of the data of Investigation II.

TABLE 5
Number Series: Minimal Distances and Mean Distance Between the Solution Patterns and the Hypothetical Knowledge Space for Investigation I and Investigation II and the Whole Group of Subjects

	Distances				d
	0	1	2	3	
Investigation I (18 subjects)	16	2	—	—	.11
Investigation II (30 subjects)	18	7	3	2	.63
Both investigations (48 subjects)	34	9	3	2	.44

TABLE 6
Number Series: Minimal Distances and Mean Distance Between the Solution Patterns and the Knowledge Space that is Based on a *Lexicographic Order* for Investigation I and Investigation II and the Whole Group of Subjects.

	Distances				d
	0	1	2	3	
Investigation I (18 subjects)	4	11	2	1	1.00
Investigation II (30 subjects)	6	12	7	5	1.37
Both investigations (48 subjects)	10	23	9	6	1.23

An alternative and "more economical" theory for the data could be stated by a lexicographic order on the problem set. For this purpose, we assume that component M_1 (recursion level) is the 'most important' component, M_2 (multiplicative factor) the second most important, and M_3 the least important component. With respect to this order, only 4 solution patterns of Investigation I and 6 patterns of Investigation II agree with a state. Table 6 provides an overview. In this case, only 12 knowledge states (consisting of 11 problems) are assumed to exist—these are about .6 % of the potential response patterns. Although the lexicographic order is much more "restrictive" than the componentwise order, these results may also be a product of the assumption concerning the importance of the components.

We assumed that a subject who is able to solve a problem, will use one particular calculation rule. This is not always realistic because, for every number series problem, alternative solutions can be found. These alternatives are frequently also plausible. The next section deals with alternative solutions.

Ambiguity of Number Series Problems

In correspondence with problem construction, the calculation rule for the series $5, 11, 21, 43, 85, 171, \ldots$? is $x_n = 2x_{n-2}+x_{n-1}$. This is problem (a_2, b_1, c_2). Obviously, the rule $x_n = 2x_{n-1} + (-1)^n$ will also provide a correct solution.

Although the subjects were told that only positive constants are to be added in the problems, we cannot exclude the possibility that a subject will use such an alternative rule. In the reported Investigation I, eleven subjects provided a correct answer, whereby eight subjects used the alternative rule as shown above.

We can see that the construction and ordering of number series problems must be based on an exact analysis of the uniqueness of the problems, especially if those are constructed from components that include principles of solution. Korossy (1998) examined the phenomenon of ambiguous number series problems with special reference to the case of linear recursive series. He developed a method, that allows the uniqueness of the solution to be determined. This method is based on the theory of linear equation systems. One of the main results of his study is that only heavy restrictions on the domains of the recursive formulas lead to less ambiguous ranges for the solutions. In the case of different rules for a problem, a generalized model using surmise systems and knowledge spaces and the respective principles for constructing these structures (see Held, 1993; Held, Chapter 4, this volume) may be appropriate.

As an overall conclusion, we can say that it is impossible to construct a number series problem that can be solved by only one rule. However it is possible to minimize the number of alternatives to a degree that allows one to work with this type of problem. Furthermore, if the manner in which an ambiguous problem has been solved is known, it may be possible to infer which of the assumed cognitive demands has been mastered by the subject.

GENERAL DISCUSSION

In this chapter, methods for the generation of ordered problem sets are introduced. Our theoretical results are motivated by the theory of knowledge spaces (Doignon & Falmagne 1985). A basic concept of this theory is the *surmise relation*, a transitive and reflexive binary relation defined on a set of problems. By this relation, a set of *knowledge states* (i. e. subsets of the problem set) is determined. Although the step from surmise relations to the more general concept of *surmise systems* is the main achievement of the theory of knowledge spaces, we restrict our considerations concerning this theory to surmise relations.

The question we are focusing on is how surmise relations can be derived from a systematically constructed set of problems. Both problem construction and problem ordering are based on *domain specific theories*, which are prerequisites for the definition of *problem components* and the establishment of *problem structures* that are derived from these components. Problem components may, for example, be operations necessary for a problem solution or subgoals during the solution process.

The methods introduced for the establishment of ordered problem sets are in principle known from elementary ordering theory and are well known in psychology: *set inclusion* and *componentwise ordering of product sets*.

We present two applications of the methods introduced for problem construction and problem ordering. The first investigation deals with the domain of *problem solving*. Chess problems are constructed on the basis of *motives*. These tactical elements of the game of chess are viewed as subgoals for the solution process that have to be detected as well as realized. A surmise relation on the problem set is established by inclusion of motive sets. The second investigation belongs to the domain of *inductive reasoning* with the solution of number series problems. Problem construction is done by product formation. The surmise relation is a result of the componentwise ordering of products. In this case, the components are parts of the *rules* that have to be found for problem solution.

Further principles for the establishment of knowledge structures that are based on problem components or skills have been developed. Lukas and Micka (1993) considered the assignment of skills to elementary chess-endgame problems. In Lukas (1997), the solution of problems on basic electricity circuits is modeled by *information systems*. This approach also focuses on *incompatibility relations* between skills. These results are also important for the definition of component-based problems as introduced in this article.

The investigations of Korossy (1993, 1997) are based on modeling *competencies* and *performances* within assessment processes. The domain under investigation is the field of *geometric constructions and calculations*. In Held (1992, 1993), knowledge spaces are derived from component-based problems on *elementary combinatorics and probability calculus*. Some of the theoretical approaches introduced there are extensions of the methods of this chapter (e. g., principles for constructing surmise systems and surmise relations). Furthermore, the assignment of “problem demands” to problem components is discussed.

Albert, Schrepp and Held (1994) provided the principle of *sequence inclusion* for ordering motive-based chess problems. This method is an extension of set inclusion that has been used for the chess experiment reported here.

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