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Component-based Construction of Surmise Relations for Chess Problems

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An approach to construct surmise relations or quasi–ordinal knowledge spaces through ordering principles is described. The ordering principles apply to the components of problems, which are considered as the basic units of knowledge necessary to solve the problems properly. We describe the three ordering principles "set inclusion", "multiset inclusion" and "sequence inclusion" and an application of these principles to the construction of surmise relations on sets of chess problems. The basic units for the construction of surmise relations in chess are the tactical elements of the game—the "motives". In terms of problem solving, these motives can be regarded as subgoals in the process of problem solving. The empirical validity of the described ordering principles is tested in two experimental investigations. The results show that the two principles "multiset inclusion" and "sequence inclusion" predict the difficulty of chess problems rather well, whereas the principle "set inclusion" is clearly insufficient in this field. The experimental investigations also demonstrate the suitability of the theory of knowledge spaces for testing psychological theories.

INTRODUCTION

A crucial problem in the theory of knowledge spaces is how to establish a knowledge space on an item set. This may be done in several ways. First, there is the possibility of querying experts. Questioning procedures have been developed by Dowling (1991, 1993), Koppen and Doignon (1990), and Koppen (1993). Another approach is to analyze response patterns as proposed by Airasian and Bart (1973), Bart and Krus (1973), and Van Leeuwe (1974). A third approach—the one used in our investigations—is to infer the knowledge space from ordering principles or skill assignments, which apply to the components of the problems. This approach is also described in Albert and Held (1994), Albert, Schrepp, and Held (1994), Doignon (1994), Held (1993), and Korossy (1993).

The ordering principles *set inclusion*, *multiset inclusion*, and *sequence inclusion*, which are described in the following section in detail, enable us to construct surmise relations on problem sets, that is they always lead to quasi-ordinal knowledge spaces. We apply these principles in order to construct surmise relations in the domain of chess.

Chess involves one of the most complex and demanding knowledge domains. Both the game of chess itself and also the construction of problems that will serve to assess knowledge concerning the tactical elements of chess require much knowledge and experience and—as we show here—some special principles.

What types of problems might prove suitable for assessing a player's knowledge of chess? For which types of problems within the large domain of chess playing is experimental exploration feasible? Because chess is so very complex we restrict considerations to problems with unique solutions. In addition, we need to be able to specify the particular knowledge of chess, that is needed to solve each problem.

Tactical chess problems fulfill these requirements. Two typical problems of this kind are shown in Fig. 1. In both positions White moves first, and the problem is to find the best moves for White. The solution of problem (a) is: 1. Ktd5+ cxd5; 2. Bg3 Qxg3 stalemate. The solution of problem (b) is: 1. Bf4 Qxf4; 2. Ktd5+ K ˜ ; 3. Ktxf4 draw.

Solving such problems requires knowledge of tactical elements, that are called "motives" in chess terminology. We try to use such motives for a classification of the problems' difficulty with respect to the other elements of the problem set. The motives considered in our investigations are "fork", "guidance", "elimination", "clearing", "promotion" and "stalemate". Definitions of these motives are given in Table 1.

We now provide an example of how a chess problem can be coded as a sequence of motives. The goal of White in problem (a) is to achieve a draw by forcing stalemate. To achieve this, White must eliminate the Knight on c3. The move 1. Ktd5+ forces the elimination (cxd5) of the Knight. The move 2. Bg3 forces the black Queen to the disadvantageous square g3 by Qxg3, and reaches

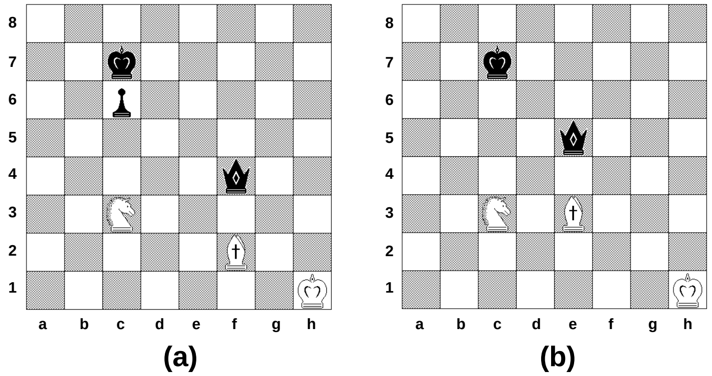

FIG. 1. Two examples for tactical chess problems.

TABLE 1
Description of Motives

Motive	*Name*	*Description*
F	Fork	One piece simultaneously attacks two opposing pieces of higher value.
G	Guidance	An opposing piece is forced to a disadvantageous square.
E	Elimination	The elimination of an own piece is forced, e.g., to achieve a stalemate.
C	Clearing	An important line or square is left, e.g., to achieve a stalemate.
T	Promotion	A pawn has to be promoted to the most suitable piece, by moving to the 8th line.
S	Stalemate	A stalemate has to be anticipated and has either to be provoked or avoided.

a draw by stalemate. Hence, the three motives “elimination”, “guidance” and “stalemate” are contained in problem (a) in this sequence. The goal in problem (b) is (by winning the black Queen) to reach a draw. The move 1. Bf4 forces the black Queen to the disadvantageous square f4 (Qxf4) and the ‘fork’ 2. Ktd5+ wins the black Queen. Therefore, the motives “guidance” and “fork” are contained in problem (b) in this sequence.

As can be seen from these examples, the same move Ktd5+ in nearly identical positions is related to different motives. In problem (a) it is based on the motive “elimination”, in problem (b) on the motive “fork”. Hence, motives are mainly considered as “standard ideas” or “subgoals in the problem solving process”. Note that we only use motives that occur as subgoals in the solution plan. For example in problem (a) of Fig. 1, the move 2. Bg3 is not based on the motive “pin” (preventing an opposing piece from moving), because “guidance”, and not “pin” is a subgoal for reaching the terminal goal “stalemate”. That is because, in this case, the Queen is not prevented from moving, but forced to move to g3.

From our point of view, knowledge or recognition of a motive means that a motive has been learned before and can be detected within a solution process. Recognizing the motives of a problem results in a solution plan. The solution process for these types of problems is therefore determined mainly by the combination and sequence of motives within a problem.

Motives are surely elements of the knowledge a person must possess if she or he is to be capable of solving the problems used in our investigations. But, there are probably other factors involved in a chess problem that also have an influence on the solution process. These factors are not the subject of our investigation; considerations regarding these factors can be found in the general discussion below.

We next introduce the theoretical principles underlying our investigations. We show how surmise relations may be established on sets of component-based chess problems. These surmise relations are inferred from psychological assumptions which are described below. These relations, the corresponding sets of knowledge states, constitute the hypotheses of our empirical investigations. The test of a hypothesis will therefore consist mainly of comparing the set of theoretically inferred knowledge states with the response patterns observed in an experiment.

THEORY

In this section, we describe three principles[1] for the component based establishment of surmise relations on sets of chess problems. These principles are based on the idea that the difficulty of a chess problem depends mainly on the motives that a subject must recognize in order to find the correct solution of the problem.

Assume a finite set M of motives. We define the *problem space* $P(M)$ as the set of all chess problems that could be characterized by the motives in M.

Principle 1 (set inclusion)

For a chess problem p let $F(p)$ be the set of motives that a subject must recognize in order to find a correct solution for p. Assume that the difficulty of a chess problem p depends only on $F(p)$, all other factors are taken to be constant for all

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[1] These principles are also reported in Albert, Schrepp and Held (1994). Principle 1 is also discussed in Albert and Held (1994).

problems under investigation. To establish a surmise relation on $P(M)$, we can therefore identify a problem $p \in P(M)$ with the set of motives $F(p) \subseteq M$. The relation $\preceq_1$ is defined for all problems $p, q \in P(M)$ by the following condition,

$$p \preceq_1 q :\Leftrightarrow F(p) \subseteq F(q).$$

This means that if a person is able to solve problem q, then she or he is also able to solve any problem p that can be solved with the knowledge of a subset of $F(q)$. From set theory it is clear that $\preceq_1$ is a quasi–order[2] and hence a surmise relation.

It is clear from chess experience that some motives are easier to recognize than others. Therefore, the motives themselves can be ordered with respect to their difficulty. We assume that the differences in the difficulty of motives can be described by a quasi-order $\sqsubseteq_M$ on M. For motives $m_i, m_j \in M$ the interpretation of $m_i \sqsubseteq_M m_j$ is "every person who is able to recognize m_i is also able to recognize m_j".

Because we want to consider also such differences in the difficulty to recognize motives, we have to generalize the set inclusion principle with respect to the relation $\sqsubseteq_M$ on M. We define a relation $\preceq_1'$ for all $p, q \in P(M)$ by,

$$p \preceq_1' q :\Leftrightarrow F(p) \subseteq' F(q),$$

where the relation $\subseteq'$ is for two subsets $F(p), F(q)$ of M characterized through the following definition.

DEFINITION 3.1 Let $\{m_1, \ldots, m_k\}, \{m_1', \ldots, m_l'\} \subseteq M$. Then $\{m_1, \ldots, m_k\} \subseteq' \{m_1', \ldots, m_l'\}$, if and only if there exists an injective[3] mapping $f : \{1, \ldots, k\} \to \{1, \ldots, l\}$ with $m_i \sqsubseteq_M m'_{f(i)}$ for all $i \in \{1, \ldots, k\}$. □

The reflexivity of $\subseteq'$ is obvious (chose f as identity). The transitivity of $\subseteq'$ follows because of the transitivity of $\sqsubseteq_M$ and the fact that the convolution of two injective mappings is also injective. Hence $\subseteq'$ is a quasi–order on the power set of M. $\subseteq'$ depends strongly on the relation $\sqsubseteq_M$ on M. This dependency is illustrated by the following example.

EXAMPLE 3.1 Let $M := \{a, b, c\}$. Fig. 2 shows the resulting quasi–order $\subseteq'$ on the power set of M for two different assumptions concerning $\sqsubseteq_M$. In one $a \sqsubseteq_M b \sqsubseteq_M c$ is assumed (all edges), in the other this assumption is dropped (solid edges only). □

Note that for the special case $m_i \sqsubseteq_M m_j \Leftrightarrow i = j$ the relations $\subseteq$ and $\subseteq'$, and therefore also the surmise relations $\preceq_1$ and $\preceq_1'$, are identical. In general, we have $p \preceq_1 q \Rightarrow p \preceq_1' q$, i.e. $\preceq_1$ is included in $\preceq_1'$.

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[2] $\preceq_1$ is not antisymmetric, i.e. not a partial order, since different problems may be represented by the same motive set.

[3] A mapping $f : \{1, \ldots, k\} \to \{1, \ldots, l\}$ is called injective if it fulfills the condition $i \neq j \Rightarrow f(i) \neq f(j)$ for all $i, j \in \{1, \ldots, k\}$.

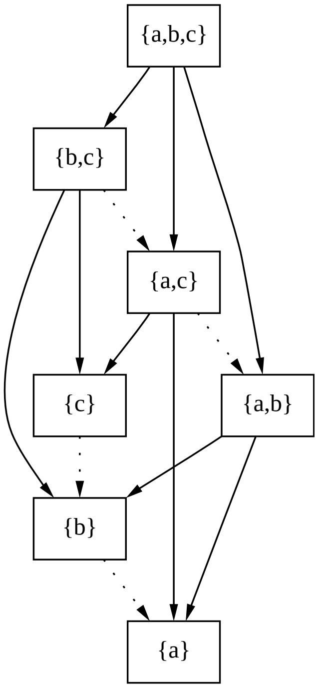

FIG. 2. Surmise relation on $P(M)$, according to "Principle 1" (set inclusion). Solid edges indicate the resulting quasi-order $\subseteq'$, if it is assumed that $x \sqsubseteq_M y \Leftrightarrow x = y$, with $x, y \in \{a, b, c\}$. Dotted edges indicate the additional pairs, if it is assumed that $a \sqsubseteq_M b \sqsubseteq_M c$.

Principle 2 (multiset inclusion)

Within Principle 1, only the occurrence of motives in the solution of a chess problem is taken into account for the construction of a surmise relation. However, it may be necessary to detect a motive more than once to solve a chess problem. So, it seems to be plausible that the multiplicity of motives occurring in the solution of a chess problem also influences its difficulty. To describe this influence we use the concept of a *multiset*. A multiset is a set with the additional property that elements can occur more than once. We write in the following $[x_1, \ldots, x_n]$ for a multiset containing the elements $x_1, \ldots, x_n$. Note that in contrast to usual sets some of the elements can be identical,so $x_i = x_j$ can occur for some $i, j \in \{1, \ldots, n\}$ with $i \neq j$. As an example for the difference between multisets and sets in the usual sense we have $\{x, x\} = \{x\}$ but $[x, x] \neq [x]$. We can define a binary relation $\subseteq_m$, called *multiset inclusion*, on the multisets by:

$$[m_1, \ldots, m_k] \subseteq_m [m'_1, \ldots, m'_l],$$

if and only if there exists an injective function $f : \{1, \ldots, k\} \to \{1, \ldots, l\}$ with $m_i = m'_{f(i)}$. Hence, a multiset X is included in a multiset Y, if and only if every element occurs at least as frequently in Y as in X. For example, we have $[x, y] \subseteq_m [x, x, y] \subseteq_m [x, x, y, y]$. It is obvious that $\subseteq_m$ is a partial order.

For a chess problem p let $G(p)$ be the multiset of motives that a subject must recognize in order to find a correct solution for p. Suppose that the difficulty of a chess problem p depends only on the multiset $G(p)$ of motives contained in the solution of p. To establish a surmise relation $\preceq_2$ on $P(M)$ it is therefore sufficient, because we can identify a problem p with $G(p)$, to define for problems $p, q \in P(M)$,

$$p \preceq_2 q :\Leftrightarrow G(p) \subseteq_m G(q).$$

As in Principle 1, we want also to consider differences in the difficulty to recognize motives. So, we have to generalize the multiset inclusion principle with respect to the relation $\sqsubseteq_M$ on the motive set M. We define a relation $\preceq'_2$ for $p, q \in P(M)$ by,

$$p \preceq'_2 q :\Leftrightarrow G(p) \subseteq'_m G(q).$$

The relation $\subseteq'_m$ is given by the following definition. Let $L(M)$ be the set of all multisets containing only elements from M.

DEFINITION 3.2 Let $[m_1, \ldots, m_k], [m'_1, \ldots, m'_l] \in L(M)$. Then $[m_1, \ldots, m_k] \subseteq'_m [m'_1, \ldots, m'_l]$, if and only if there exists an injective function $f : \{1, \ldots, k\} \to \{1, \ldots, l\}$ with $m_i \sqsubseteq_M m'_{f(i)}$ for all $i \in \{1, \ldots, k\}$. □

The reflexivity and transitivity of $\subseteq'_m$ follows, as in Principle 1, from the transitivity of $\sqsubseteq_M$ and the fact that the convolution of injective functions is also an injective function. Note that $\subseteq'_m$ depends strongly on the assumed quasi–order $\sqsubseteq_M$ on the motive set M. This dependency is illustrated by the following example.

EXAMPLE 3.2 Let $M := \{a, b\}$. Fig. 3 shows the resulting quasi-order $\subseteq'_m$ on the set of all multisets containing at least two elements of M for two different assumptions concerning $\sqsubseteq_M$. In one $a \sqsubseteq_M b$ is assumed (all edges), in the other this assumption is dropped (solid edges only). □

Note that for the special case $m_i \sqsubseteq_M m_j :\Leftrightarrow i = j$ we have $\subseteq'_m = \subseteq_m$, the relation $\subseteq'_m$ is identical with the multiset inclusion $\subseteq_m$, which implies trivially that the surmise relations $\preceq_2$ and $\preceq'_2$ are also in this special case identical. In general, we have $p \preceq_2 q \Rightarrow p \preceq'_2 q$, i.e. $\preceq_2$ is included in $\preceq'_2$.

Principle 3 (sequence inclusion)

In Principles 1 and 2, the order in which the motives occur in the solution of a chess problem does not affect the resulting surmise relation. Ignoring this order

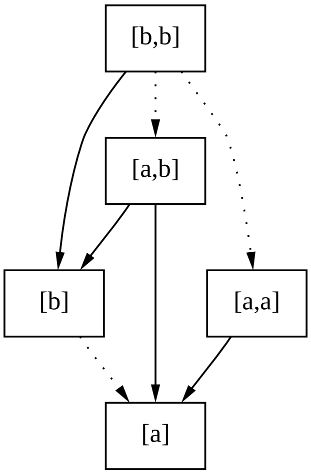

FIG. 3. Surmise relation on $P(M)$, accordingly to “Principle 2” (multiset inclusion). Solid edges indicate the resulting quasi-order $\subseteq'_m$, if it is assumed that $x \sqsubseteq_M y \Leftrightarrow x = y$, with $x, y \in \{a, b\}$. Dotted edges indicate the additional pairs, if it is assumed that $x \sqsubseteq_M y \Leftrightarrow x = y \vee (x = a \wedge y = b)$, with $x, y \in \{a, b\}$.

may be problematical for the analysis of chess problem solving, because of forward and backward search strategies in problem solving. In Principle 3, the order in which the motives occur in the solution of a problem plays a central role in the construction of the surmise relation on $P(M)$. The central idea of Principle 3 is that a problem a is more difficult than a problem b if the sequence of motives that must be recognized by a subject to solve a includes the sequence of motives that must be recognized to solve b.

For a chess problem $p \in P(M)$, let $H(p)$ be the ordered tuple of motives that occur in the solution of the problem. $H(p) = (m_1, \ldots, m_k)$ means that the first motive which occurs is m_1, the second is m_2, etc. Note that one motive can occur more than once within $H(p)$. In this case, the multiplicity of motives is also covered, for example, $m_i = m_j$ for $i \neq j$.

The order in which the motives occur in the solution of a problem[4] is very important for Principle 3. However, it will be clear from our ordering principle that the resulting surmise relation will not change if the order $(m_1, \ldots, m_k)$ is, *for all* problems, inverted to $(m_k, \ldots, m_1)$. We define,

$$M_k := \{(m_1, \ldots, m_k) \mid m_1, \ldots, m_k \in M\},$$
$$M_{\mathbb{N}} := \textstyle\bigcup_{k \in \mathbb{N}} M_k.$$

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[4] The order, in which the motives are recognized by a subject, is not necessarily identical with the order in the solution of a problem, and in the solution plan, respectively.

M_k is the set of all ordered k-tuples of motives from M and $M_{\mathbb{N}}$ is the set of all ordered tuples of motives from M. We define a binary relation on $M_{\mathbb{N}}$ by,

$$(m_1, \ldots, m_k) \sqsubseteq (m'_1, \ldots, m'_l),$$

if and only if there exists a strictly increasing[5] function $f : \{1, \ldots, k\} \to \{1, \ldots, l\}$ with $m_i = m'_{f(i)}$. Hence $(m_1, \ldots, m_k) \sqsubseteq (m'_1, \ldots, m'_l)$, if and only if $(m_1, \ldots, m_k)$ can be obtained by deleting motives from $(m'_1, \ldots, m'_l)$. For example, we have $(x, z) \sqsubseteq (x, y, z)$ but $(x, z) \not\sqsubseteq (z, y, x)$ because we can get (x, z) by deleting y from (x, y, z) but no deletion of an element of (z, y, x) can yield (x, z).

Suppose that under suitable conditions the difficulty of a chess problem $p \in P(M)$ depends only on $H(p)$. To establish a surmise relation $\preceq_3$ on $P(M)$, we identify a problem p with $H(p)$. It is then sufficient to define for problems $p, q \in P(M)$,

$$p \preceq_3 q :\Leftrightarrow H(p) \sqsubseteq H(q).$$

We extend the sequence inclusion principle in order to consider also differences in the difficulty to recognize motives. We define a relation $\preceq'_3$ for $p, q \in P(M)$ by

$$p \preceq'_3 q :\Leftrightarrow H(p) \sqsubseteq' H(q),$$

where the relation $\sqsubseteq'$ is given through the definition below.

DEFINITION 3.3 Let $(m_1, \ldots, m_k), (m'_1, \ldots, m'_l) \in M_{\mathbb{N}}$. Then $(m_1, \ldots, m_k) \sqsubseteq' (m'_1, \ldots, m'_l)$, if and only if there exists a function $f : \{1, \ldots, k\} \to \{1, \ldots, l\}$ that fulfills the following conditions:

1. $\forall i, j \in \{1, \ldots, k\} \quad (i < j \to f(i) < f(j))$,
2. $\forall j \in \{1, \ldots, k\} \quad (m_j \sqsubseteq_M m'_{f(j)})$. □

The reflexivity of $\sqsubseteq'$ is obvious (choose the identity for f). The transitivity of $\sqsubseteq'$ follows because of the transitivity of $\sqsubseteq_M$ and the fact that the convolution of two monotonic increasing functions also has this property. Hence, $\sqsubseteq'$ is a quasi-order, that depends on the relation $\sqsubseteq_M$ on M. The following example illustrates this dependency.

EXAMPLE 3.3 Let $M := \{a, b\}$. Fig. 4 shows the resulting quasi-order $\sqsubseteq'$ on $M_1 \cup M_2 \subseteq P(M)$ for two different assumptions concerning $\sqsubseteq_M$. In one $a \sqsubseteq_M b$ is assumed (all edges), in the other this assumption is dropped (solid edges only). □

For the special case $m_i \sqsubseteq_M m_j :\Leftrightarrow i = j$ we have $\sqsubseteq = \sqsubseteq'$ and therefore $\preceq_3 = \preceq'_3$. In general $p \preceq_3 q \Rightarrow p \preceq'_3 q$, so $\preceq_3$ is included in $\preceq'_3$.

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[5] A function $f : \{1, \ldots, k\} \to \{1, \ldots, l\}$ with $i < j \Rightarrow f(i) < f(j)$ is called strictly increasing.

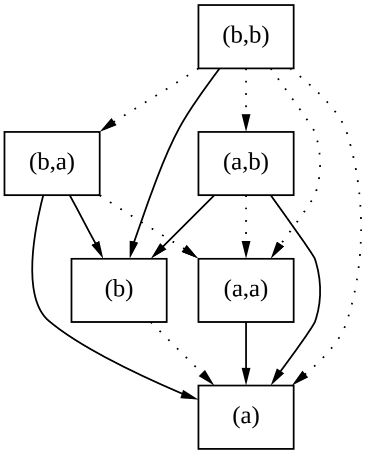

FIG. 4. Surmise relation on $P(M)$, accordingly to “Principle 3” (sequence inclusion). Solid edges indicate the resulting quasi-order $\sqsubseteq$, if it is assumed that $x \sqsubseteq_M y \Leftrightarrow x = y$, with $x, y \in \{a, b\}$. Dotted edges indicate the additional pairs, if it is assumed that $x \sqsubseteq_M y \Leftrightarrow x = y \vee (x = a \wedge y = b)$, with $x, y \in \{a, b\}$.

Comparison of the Principles

A comparison of the surmise relations $\preceq_1'$, $\preceq_2'$, and $\preceq_3'$ resulting from our three ordering principles shows that $\preceq_1'$ is stricter[6] than $\preceq_2'$ and that $\preceq_2'$ is stricter than $\preceq_3'$. We summarize and prove this dependency in the following theorem.

Theorem: Let M be a set of motives and $p, q \in P(M)$. Then we have

1. $p \preceq_2' q \Rightarrow p \preceq_1' q$
2. $p \preceq_3' q \Rightarrow p \preceq_2' q$

Proof: First, we show the implication $p \preceq_2' q \Rightarrow p \preceq_1' q$. Let $G(p) = [m_1, \ldots, m_k]$ and $G(q) = [m_1', \ldots, m_l']$ be the multisets of motives that must be recognized to solve the problems p and q. $p \preceq_2' q$ implies by definition the existence of an injective function $f : \{1, \ldots, k\} \rightarrow \{1, \ldots, l\}$ with $m_i \sqsubseteq_M m_{f(i)}'$. Because we have $F(p) = \{m_1, \ldots, m_k\}$ and $F(q) = \{m_1', \ldots, m_l'\}$ this trivially implies $F(p) \subseteq' F(q)$. Hence, the implication $p \preceq_2' q \Rightarrow p \preceq_1' q$ follows directly.
Second, we show $p \preceq_3' q \Rightarrow p \preceq_2' q$. Let $H(p) = (m_1, \ldots, m_k)$ and $H(q) = (m_1', \ldots, m_l')$ be the sequences of motives occurring in the solutions of the problems p and q. $p \preceq_3' q$ implies the existence of a function $f : \{1, \ldots, k\} \rightarrow \{1, \ldots, l\}$ with the properties $\forall i, j \in \{1, \ldots, k\}\ (i < j \Rightarrow f(i) < f(j))$ and

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[6] We call a relation R to be stricter than a relation R' if R' is included in R($R' \subseteq R$). This is equivalent to the fact that for every x and y the implication $xRy \Rightarrow xR'y$ is true.

$\forall j \in \{1, \ldots, k\}\ (m_j \sqsubseteq_M m_{f(j)})$. Because every strictly increasing function is also injective, this directly implies $p \preceq_2' q$. $\square$

KNOWLEDGE SPACES FOR CHESS PROBLEMS

For an empirical test of the three ordering principles described above, a set of 16 tactical chess problems has been constructed. For the solution of each of these problems only the motives shown in Table 1 have to be detected. Therefore the motive set M is given through $M = \{F, G, E, C, T, S\}$. Each problem contains four motives at the most. The number of moves necessary for the solution of the problems ranges from one to four, although one move does not necessarily represent one motive. In all problems White moves first and has to find the optimal moves in the given position. The optimal moves do not necessarily lead to a mate. Forcing a draw by stalemate or reaching a winning position can also be optimal solutions. A complete list of the 16 problems, their characterization as tuples of motives and the solutions are provided in Table 2.

In the following, we characterize chess problems by the ordered tuples of motives occurring in their solution. From this characterization as motive tuples, the corresponding characterizations as multisets or sets of motives can easily be derived.

Remember that the surmise relations $\preceq_1'$, $\preceq_2'$, and $\preceq_3'$ derived from the principles “set inclusion”, “multiset inclusion” and “sequence inclusion” depend to a great extent on the assumed quasi–order $\sqsubseteq_M$ on the motive set. As mentioned already for the special case $x \sqsubseteq_M y \Leftrightarrow x = y$ for all $x, y \in \{F, G, E, T, C, S\}$, where is assumed that none of the motives can be detected more easily than other motives, we have $\preceq_1 = \preceq_1'$, $\preceq_2 = \preceq_2'$, and $\preceq_3 = \preceq_3'$.

For our ordering principles, we distinguish in the following between two types of different hypothetical problem structures. If we assume that none of the motives can be recognized more easily than other motives, ($m_i \sqsubseteq_M m_j \Leftrightarrow i = j$), the surmise relations $\preceq_1$, $\preceq_2$, and $\preceq_3$ consist of two nonconnected structures. If we assume that the motive fork can be detected more easily than the motives S, E, C, G, T formalized by $m_i \sqsubseteq_M m_j \Leftrightarrow i = j \vee (m_i = F \wedge m_j \in \{S, E, C, G, T\})$, surmise relations $\preceq_1'$, $\preceq_2'$, and $\preceq_3'$ with additional relational dependencies between problems result.

The surmise relations $\preceq_1$ and $\preceq_1'$ on the problem set are depicted in Fig. 5 as a Hasse–Diagram. The surmise relations $\preceq_2$ and $\preceq_2'$ are depicted in Fig. 6, and the surmise relations $\preceq_3$ and $\preceq_3'$ are depicted in Fig. 7.

Note that the relations $\preceq_2$ and $\preceq_3$ differ only in three relational dependencies. For $\preceq_2$ we have $(G, E, S) \preceq_2 (E, G, S)$, $(E, G, S) \preceq_2 (G, E, S)$, and $(G, E, S) \preceq_2 (E, E, G, S)$, whereas for $\preceq_3$, these relational dependencies do not hold, that means, $(G, E, S) \not\preceq_3 (E, G, S), (E, G, S) \not\preceq_3 (G, E, S)$, and $(G, E, S) \not\preceq_3 (E, E, G, S)$. Because of this small difference it seems to be hard to decide empir-

TABLE 2

Complete List of Chess Problems for Experiment 2

Type	*Position*	*Solution*	*Alternative*
(S)	*White*: Ka1 Qf5 *Black*: Kh8 Pg6	1. Q arbitrary except of Qg6:,Qh5	
(G, S)	*White*: Kh1 Bf2 *Black*: Kc7 Qe5	1. Bg3 Qg3: stale-mate	
(E, G, S)	*White*: Kh1 Bf2 Ktc3 *Black*: Kc7 Qf4 Pc6	1. Ktd5+ cd5: 2. Bg3 Qg3: stale-mate	
(E, E, G, S)	*White*: Ka1 Bc2 Ktf3,d7 *Black*: Kf7 Qc4 Pf6,d6	1. Kte5+ de5:/fe5: 2. Kte5:+ fe5:/de5: 3. Bb3 Qb3: stale-mate	
(C, S)	*White*: Ka3 Pb2 *Black*: Kc5 Rh2 Pa4,b5,c4	1. b4+ ab3:ep/cb3:ep stale-mate	
(G, C, S)	*White*: Kh3 Bg4 Pg2 *Black*: Kg6 Rc2 Ph4,g5,f4	1. Bf5+ Kf5: 2. g4+ hg3:ep/fg3:ep stale-mate	
(T, S)	*White*: Kh5 Pf7 *Black*: Kh7	1. f8R win	
(G, E, S)	*White*: Ka1 Bc2 Kte2 *Black*: Ke6 Qc4 Pe5	1. Bb3 Qb3: 2. Ktd4+ ed4: stale-mate	
(F)	*White*: Kb2 Ktf3 *Black*: Kf7 Qc6	1. Kte5+ draw	
(G, F)	*White*: Kh1 Be3 Ktc3 *Black*: Kc7 Qe5	1. Bf4 Qf4: 2. Ktd5+ draw	
(G, F, F)	*White*: Kg1 Bc6 Kte4,g4 *Black*: Kg8 Qe6 Pg7	1. Bd5 Qd5: 2. Ktf6+ gf6: 3. Ktf6:+ draw	
(G, G, F, F)	*White*: Kb1 Bg3 Kth4,e7 Pf7,c2 *Black*: Kh8 Qf6 Bg7 Ph7	1. f8Q(R)+ Bf8: 2. Be5 Qe5: 3. Ktg6+ hg6: 4. Ktg6:+ draw	1. Ktg6+ hg6: 2.f8Q+ Bf8: 3. Be5 Qe5: 4. Ktg6:+ draw
(G, G, F)	*White*: Kb1 Bg3 Kth4 Pf7,e2 *Black*: Kh8 Qf6 Bg7	1. f8Q(R)+ Bf8: 2. Be5 Qe5: 3. Ktg6+ draw	
(F, F)	*White*: Kb2 Kte3,c3 *Black*: Kc7 Qf4 Pe6	1. Ktd5+ ed5: 2. Ktd5:+ draw	
(T, F)	*White*: Kc1 Pf7,b2 *Black*: Kh7 Qd7	1. f8Kt+ win	
(T, F, F)	*White*: Kb1 Kte4 Pf7 *Black*: Kh7 Qd7 Bg7	1. f8Kt+ Bf8: 2. Ktf6+ draw	1.Ktf6+ Bf6: 2.f8Kt+ draw

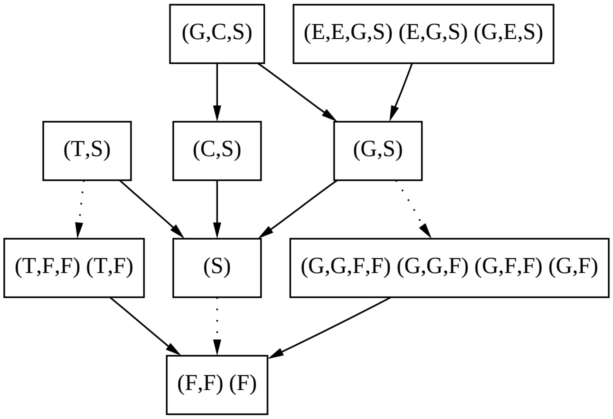

FIG. 5. Solid edges indicate the resulting surmise relation $\preceq_1$ (set inclusion), assuming that $x \sqsubseteq_M y \Leftrightarrow x = y$, with $x, y \in \{F, G, E, T, C, S\}$. All edges indicate the surmise relation $\preceq_1'$, assuming that $x \sqsubseteq_M y \Leftrightarrow x = y \vee (x = F \wedge y \in \{G, E, T, C, S\})$.

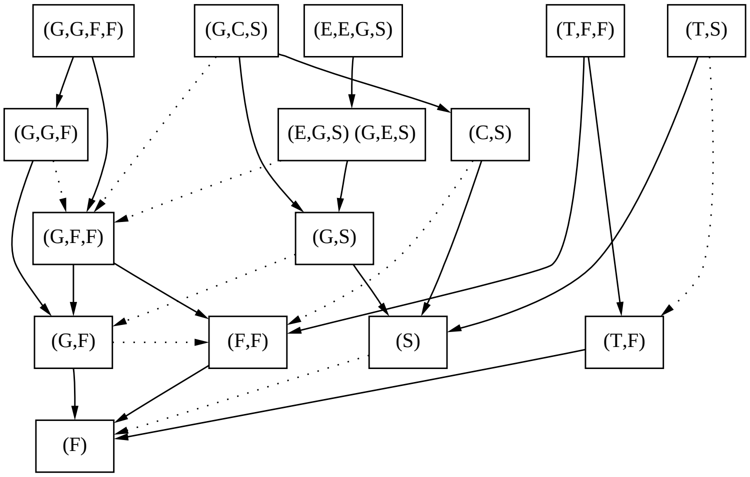

FIG. 6. Solid edges indicate the resulting surmise relation $\preceq_2$ (multiset inclusion). It is assumed that $x \sqsubseteq_M y \Leftrightarrow x = y$, with $x, y \in \{F, G, E, T, C, S\}$. All edges indicate the surmise relation $\preceq_2'$. Here it is assumed that $x \sqsubseteq_M y \Leftrightarrow x = y \vee (x = F \wedge y \in \{G, E, T, C, S\})$.

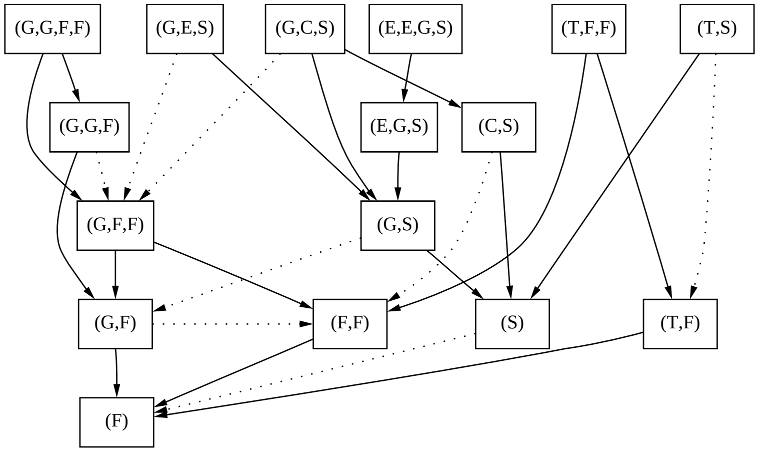

FIG. 7. Solid edges indicate the resulting surmise relation $\preceq_3$ (sequence inclusion). It is assumed that $x \sqsubseteq_M y \Leftrightarrow x = y$, with $x, y \in \{F, G, E, T, C, S\}$. All edges indicate the surmise relation $\preceq_3'$. Here it is assumed that $x \sqsubseteq_M y \Leftrightarrow x = y \vee (x = F \wedge y \in \{G, E, T, C, S\})$.

ically between Principle 2 and Principle 3 with respect to the selected problem set.

We have to mention here that it will not be easy to construct a problem set for which the surmise relations $\preceq_2$ and $\preceq_3$ show a greater difference. The reason is that the motive sequences in solutions of chess problems can not be varied arbitrarily. The motive “stalemate”, for example, can only occur at the end of such a motive sequence, whereas the motive “elimination” cannot occur as last motive in a motive sequence. An additional restriction is that some motives can occur only in connection with others. For example, the motive “elimination” can only occur in connection with “stalemate”.

The knowledge spaces corresponding to the surmise relations $\preceq_1$,$\preceq_1'$, $\preceq_2$, $\preceq_2'$, $\preceq_3$, and $\preceq_3'$ are indicated in the following as $\mathcal{K}_1$,$\mathcal{K}_1'$, $\mathcal{K}_2$, $\mathcal{K}_2'$, $\mathcal{K}_3$, and $\mathcal{K}_3'$.

EXPERIMENT I

To test the validity of our ordering principles empirically[7], we compare the quasi–ordinal knowledge spaces $\mathcal{K}_1$,$\mathcal{K}_1'$, $\mathcal{K}_2$, $\mathcal{K}_2'$, $\mathcal{K}_3$, and $\mathcal{K}_3'$ derived from these ordering principles with the response patterns of 46 subjects, that worked on the 16 problems described in Table 2. The investigation was conducted at the Psycho-

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[7] The data of Experiment I and their analysis concerning “Principle 1” and “Principle 3” are already published in Albert, Schrepp and Held (1994).

logical Institute of the University of Heidelberg.

Method

The experiment was conducted with 37 male and 9 female subjects. Their ages ranged from 14 to 71 years. All of them were familiar with the basic rules of chess. The subjects were recruited through an announcement in the local newspaper. For taking part in the investigation each subject was paid DM 12,–. The experiment was conducted in an experimental room. The problems were presented on a computer screen (SUN–3 workstation with monochrome monitor) and the subjects were required to make their moves using a mouse.

A run of the experiment consists of two main phases: a training phase and an experimental phase. The training phase allows the subjects to learn and practise moving the pieces on the computer chess board. On completion of the training phase, the experimental phase, starts with three very simple "dummy problems", that are not relevant for further analysis. After the presentation of these initial problems was finished, the 16 problems described in the last section were presented in random order.

The subject had 90 seconds to work out a solution plan to a problem. After that, each move had to be made within 30 seconds. If this time is exceeded, a message is displayed requesting the subjects to draw faster. The time limitations were chosen to preclude two kinds of strategy on the part of the subjects: (1) to prevent any subject moving too soon without having thought sufficiently about the solution to the problem, and (2) to prevent the subject continuing to work out a solution while making his or her moves.

After each move by the subject, the computer either made a suitable reply (if the subject's move was correct), or the game was terminated (if either the problem was completed or the subject's move was wrong). At no time do subjects receive feedback concerning the correctness of their answers.

Results

An overview about individual results is provided in Table 3.

First, how well do the data fit our knowledge spaces? Table 4 provides the number of states (elements of the corresponding knowledge space), the number of subjects with response patterns congruent with a state, the number of response patterns not congruent with any state, as well as the number of *different* congruent and not congruent response patterns within the data. Table 5 presents the symmetric distances [8] between response patterns and closest states.

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[8] The symmetric distance d between two sets A and B is defined as follows: $d(A, B) :=\mid A \triangle B \mid$, where $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

TABLE 3
Individual Results in Experiment I

Subjects	*Distances*					*Problems*															
	$\mathcal{K}_1$	$\mathcal{K}_2$	$\mathcal{K}'_2$	$\mathcal{K}_3$	$\mathcal{K}'_3$	S	GS	EGS	EEGS	CS	GCS	TS	GES	F	GF	GFF	GGFF	GGF	FF	TF	TFF
1,14	1	0	0	0	0	+	+	+	+	+	+	+	+	+	+	+	–	+	+	+	+
2,18,30,37	0	0	0	0	0	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
3	3	0	0	0	0	+	+	–	–	–	–	+	–	+	+	+	–	–	+	+	–
4	4	2	3	1	2	+	+	+	+	+	+	+	–	+	+	–	+	–	+	+	–
5	1	1	1	1	1	+	–	–	–	–	–	+	–	+	–	+	–	–	+	+	+
6	3	2	2	2	2	+	–	–	+	–	–	+	+	+	+	–	–	–	+	+	+
7	3	3	4	3	4	+	+	–	–	–	–	–	–	–	–	–	+	–	+	–	+
8	0	0	1	0	1	+	–	–	–	–	–	+	–	+	–	–	–	–	+	–	–
9,21,45	1	0	0	0	0	–	–	–	–	–	–	–	–	+	–	–	–	–	–	–	–
10	1	0	0	0	0	+	+	+	+	–	–	+	–	+	+	+	+	+	+	+	+
11	1	0	0	0	0	–	–	–	–	–	–	–	–	+	+	–	–	–	+	+	+
12	2	0	0	0	0	+	–	–	–	–	–	+	–	+	+	+	–	–	+	+	+
13,34	1	0	0	0	0	+	–	–	–	+	–	+	–	+	–	–	–	–	+	+	–
15	1	0	0	0	0	+	+	+	+	–	–	+	+	+	+	+	–	+	+	+	+
16	1	1	2	1	2	+	–	–	–	–	–	–	–	–	–	–	–	–	–	+	–
17	2	0	0	0	0	+	+	+	–	+	+	–	+	+	+	+	–	+	+	+	+
19	2	2	2	2	2	–	–	–	–	+	–	–	–	–	–	–	–	–	–	+	–
20	2	0	1	0	1	–	–	–	–	–	–	–	–	+	–	–	–	–	–	+	–
22	0	0	1	0	1	+	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
23	3	1	1	1	1	+	–	–	+	–	–	–	–	+	+	+	–	–	+	–	–
24	2	1	2	1	2	–	–	–	–	+	–	+	–	+	–	–	–	–	–	–	–
25	0	0	0	0	0	+	–	–	–	–	–	+	–	+	–	–	–	–	+	+	+
26	1	0	0	0	0	–	–	–	–	–	–	–	–	+	–	–	–	–	+	+	–
27,38	1	1	1	1	1	–	–	–	–	+	–	–	–	+	–	–	–	–	+	–	–
28,41	2	1	1	1	1	–	–	–	–	+	–	–	–	+	–	–	–	–	–	–	–
29	2	2	2	2	2	+	–	–	–	+	+	–	–	+	–	–	–	–	–	+	+
31	2	0	1	0	1	+	+	+	+	+	–	–	+	+	+	–	–	–	+	+	–
32	1	0	0	0	0	+	–	–	–	+	–	–	–	+	–	–	–	–	+	+	–
33	2	2	2	1	1	+	+	+	+	–	–	+	–	+	+	–	+	+	+	+	+
35	4	1	2	1	2	+	+	+	–	–	–	+	+	+	+	–	+	–	+	+	–
36,42	1	0	0	0	0	+	–	–	–	–	–	–	–	+	–	–	–	–	+	+	–
39	1	1	1	1	1	–	–	–	–	–	–	–	–	–	–	–	–	–	+	–	–
40	1	0	0	0	0	+	–	–	–	–	–	–	–	+	–	–	–	–	–	–	–
43	2	1	1	1	1	–	–	–	–	+	–	+	–	+	–	–	–	–	+	+	–
44	2	0	1	0	1	+	–	–	–	–	–	–	–	+	–	–	–	–	–	+	–
46	2	1	1	1	1	–	–	–	–	+	–	–	–	+	–	–	–	–	+	+	–

A “+” indicates that the problem was solved by the corresponding subject, while a “–” indicates that this was not the case.

TABLE 4
Possible States and Response Patterns for Experiment I

Knowl. space	*Number of states*	*Number of congruent patterns*	*Number of non congruent patterns*	*Number of different congruent patterns*	*Number of different non congruent patterns*
$\mathcal{K}_1$	85	7	39	4	32
$\mathcal{K}'_1$	35	5	41	2	34
$\mathcal{K}_2$	575	27	19	20	16
$\mathcal{K}'_2$	213	22	24	15	21
$\mathcal{K}_3$	1025	28	18	20	16
$\mathcal{K}'_3$	368	23	23	15	21

TABLE 5
Frequency of Distances Between Response Patterns and Closest States for Experiment I

Knowl. space	*Distances*					*Average Distance*
	0	*1*	*2*	*3*	*4*	
$\mathcal{K}_1$	7	20	13	4	2	1.43
$\mathcal{K}'_1$	5	21	12	4	4	1.59
$\mathcal{K}_2$	27	13	5	1	0	0.57
$\mathcal{K}'_2$	22	15	7	1	1	0.78
$\mathcal{K}_3$	28	14	3	1	0	0.50
$\mathcal{K}'_3$	23	15	7	0	1	0.72

Number of subjects who solved all problems: 4
Number of subjects who solved no problem: 0

Discussion

The results of the experiment show that the principle "set inclusion" seems to be less adequate for the ordering of chess problems concerning their difficulty than the principles "multiset inclusion" and "sequence inclusion". As expected, it seems to be impossible to decide on the basis of this results between "multiset inclusion" and "sequence inclusion".

Despite the fact that the quasi-ordinal knowledge spaces $\mathcal{K}_1$,$\mathcal{K}'_1$, $\mathcal{K}_2$, $\mathcal{K}'_2$, $\mathcal{K}_3$, and $\mathcal{K}'_3$ are assumed to be purely deterministic, in comparing them with response patterns we have to consider influences like "lucky guesses" or "careless errors" (see Falmagne & Doignon, 1988) [9].

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[9] The probability of lucky guesses and careless errors must be very low. If we assume for instance

Table 5 shows that for the quasi-ordinal knowledge spaces derived from Principles 2 and 3 the symmetric distances between response patterns and closest states are rather small. So most of the deviations between response patterns and the hypothetical knowledge spaces derived from these principles may be explained by such influences like lucky guesses or careless errors. Therefore, these principles seem to be adequate for our set of chess problems.

In analyzing the results, special attention should be paid to the proportions between the number of states and the number of nonstates (i.e. the subsets of the problem set that do not belong to the corresponding knowledge space), on the one hand, and the proportions between the number of response patterns congruent with states and the number of response patterns not congruent with states, on the other. The best fitting space $\mathcal{K}_3$ contains only 1.6 % of the theoretically possible response patterns, while it contains 61 % of the observed response patterns.

The additional assumption that the motive "fork" can be recognized more easily than all of the other motives leads for both principles, "multiset inclusion" and "sequence inclusion" to a reduction in the number of hypothetical knowledge states. It therefore comes as no surprise that this assumption leads to a larger number of response patterns which do not agree with the quasi-ordinal knowledge spaces derived from these principles. Most of them, however, have a small symmetric distance.

In comparing our ordering principles with respect to their adequacy to describe the difficulty of chess problems we have to deal with the problem that the quasi-ordinal knowledge spaces $\mathcal{K}_1$,$\mathcal{K}_1'$, $\mathcal{K}_2$, $\mathcal{K}_2'$, $\mathcal{K}_3$, and $\mathcal{K}_3'$ differ extremely in size. Because both the average symmetric distance and the number of congruent response patterns depend on the size of a knowledge space, these numbers can not be interpreted directly. An approach to solve this problem is presented in the general discussion.

EXPERIMENT II

In a replication of our first experiment, which was conducted at the University of Graz (Austria), the response patterns of 46 subjects were compared with the quasi-ordinal knowledge spaces derived from our ordering principles.

Method

The experiment was conducted with 38 male and 8 female subjects. All of them were familiar with the basic rules of chess. 21 subjects were members of chess clubs, whereas the rest were pupils or students of the University of Graz, and were

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that the probability of both careless errors and lucky guesses is 0.05, then the probability that the response pattern of the subject is equal to the subject's knowledge state is $0.95^{16} \approx 0.44$.

pure hobby-players. Therefore, the chess-playing ability of the subjects varied over a wide range. The subjects were not paid for taking part in the investigation. The ages of the subjects ranged from 6 to 64 years. The experiment was conducted partly in a chess club, a school, and in a chess cafe. The problems were presented on a laptop-computer. The laptop was operated by the experimenter, so the subjects made their moves verbally and the experimenter entered these moves into the computer.

The experiment consists, similar to Experiment I, of two phases, a instructional phase and an experimental phase. In the instructional phase the experimental setting and the way the subjects can give their solutions were explained. After the instructional phase was finished the experimental phase started with three simple "dummy problems" (identical with the dummy problems used in Experiment I), which are not relevant for further analysis. Then, the 16 problems described in Table 2 were presented in random order.

The time limitations are similar to Experiment I, the subject had 90 seconds to work out a solution plan and, after that, each move had to be made within 30 seconds. If a subject exceeds the time limitations, the experimenter requires the subject verbally to make a move. After each move of the subject, a suitable reply was shown on the screen (if the subject's move was correct), or the game was terminated (if either the problem was completed or the subjects move was wrong). At no time do the subjects receive feedback concerning the correctness of their answers.

Results

An overview about individual results is given in Table 6.

In Table 7, the number of states, the numbers of congruent and noncongruent response patterns and the number of different congruent and noncongruent response patterns is shown. Table 8 provides an overview about the observed symmetric distances between response patterns and closest states.

Discussion

A comparison of these results with the results of Experiment I show that they are nearly identical. The differences in the experimental setting had almost no influence on the results. Another remarkable point is that in Experiment II, 21 of the 46 subjects were members of chess clubs, whereas in Experiment I almost all subjects were pure hobby players. This implies that the characterization of the problems' difficulty, which is given through our surmise relations, is valid over a wide range of chess playing ability.

As in Experiment I, the principle "set inclusion" seems to be less adequate for the ordering of chess problems than the principles "multiset inclusion" and

TABLE 6
Individual Results in Experiment II

Subjects	*Distances*					*Problems*															
	$\mathcal{K}_1$	$\mathcal{K}_2$	$\mathcal{K}'_2$	$\mathcal{K}_3$	$\mathcal{K}'_3$	S	GS	EGS	EEGS	CS	GCS	TS	GES	F	GF	GFF	GGFF	GGF	FF	TF	TFF
1	2	1	1	1	1	–	–	–	–	–	–	–	+	+	–	–	–	–	+	+	–
2	2	0	0	0	0	+	+	–	–	–	–	–	–	+	+	+	–	–	+	–	–
3,32	1	0	0	0	0	+	–	–	–	–	–	–	–	+	–	–	–	–	–	–	–
4,36	1	0	0	0	0	–	–	–	–	–	–	–	–	+	–	–	–	–	–	–	–
5	1	1	1	1	1	+	+	+	+	–	–	+	+	+	+	+	+	+	–	+	+
6,16,25,29	0	0	0	0	0	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
7	3	0	0	0	0	+	+	+	–	–	–	+	+	+	+	+	–	–	+	+	+
8	3	1	1	1	1	+	+	–	+	–	–	+	+	+	+	+	–	–	+	+	+
9	2	1	1	1	1	+	–	–	–	–	–	–	–	+	–	+	–	–	–	–	–
10	1	0	0	0	0	–	–	–	–	–	–	–	–	+	–	–	–	–	–	–	–
11	2	1	3	1	3	+	–	–	–	+	–	+	–	+	–	–	–	+	–	–	–
12	2	2	2	2	2	+	+	+	+	–	+	+	+	+	+	+	+	–	+	+	+
13	2	1	2	0	2	+	+	–	–	–	–	+	+	+	–	–	–	–	+	+	–
14	2	1	1	0	1	+	+	–	–	–	–	+	+	+	+	–	–	–	+	+	+
15	1	1	1	1	1	+	+	+	+	–	–	+	+	+	+	+	+	–	+	+	+
17	1	0	0	0	0	–	–	–	–	+	–	–	–	+	–	–	–	–	–	–	–
18	2	0	1	0	1	+	–	–	–	–	–	+	–	+	–	–	–	–	–	+	–
19	0	0	0	0	0	+	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
20	0	0	0	0	0	+	–	–	–	+	–	+	–	+	–	–	–	–	+	+	+
21	2	1	1	1	1	+	+	+	+	–	+	+	+	+	+	+	–	+	+	+	+
22	3	0	0	0	0	+	+	+	–	+	+	–	+	+	+	+	–	–	+	+	+
23	4	4	4	3	3	+	+	–	–	–	+	+	+	+	+	+	+	–	+	–	+
24	2	1	1	1	1	+	+	+	–	–	–	+	+	+	+	+	+	–	+	+	+
26	2	2	2	2	2	+	–	–	–	–	–	–	+	+	+	+	+	–	+	+	+
27	1	0	0	0	0	+	+	+	–	–	–	+	+	+	+	+	+	+	+	+	+
28	1	1	1	1	1	+	+	+	+	–	+	+	+	+	+	+	+	+	+	+	+
30	3	1	2	1	2	+	–	–	–	–	+	–	–	+	–	–	–	–	–	+	–
31	3	1	2	1	2	–	–	–	–	+	–	–	–	+	–	–	–	–	–	+	–
33	1	1	1	1	1	–	–	–	–	–	–	–	–	+	–	–	–	–	+	–	+
34	1	0	0	0	0	+	–	–	–	–	–	–	–	+	–	–	–	–	+	+	–
35	3	3	3	2	3	+	–	–	–	+	+	+	+	+	–	–	–	–	–	+	+
37	1	0	0	0	0	+	–	–	–	–	–	+	–	+	–	–	–	–	+	+	–
38	1	1	2	1	2	+	+	–	–	–	–	–	–	+	–	–	–	–	+	–	+
39	2	1	1	1	1	+	–	–	–	+	–	+	+	+	–	–	–	–	+	+	–
40	1	0	0	0	0	+	–	–	–	–	–	+	–	+	+	+	–	+	+	+	+
41	1	1	1	1	1	–	–	–	–	–	–	+	–	+	–	–	–	–	+	+	+
42	3	1	2	1	2	+	–	–	–	–	–	+	+	+	–	–	–	–	–	+	–
43	1	0	0	0	0	–	–	–	–	–	–	–	–	+	–	–	–	–	+	+	–
44	0	0	0	0	0	+	–	–	–	–	–	–	–	+	–	–	–	–	+	–	–
45	1	0	0	0	0	+	+	–	–	–	–	–	–	+	+	–	–	–	+	–	–
46	1	0	1	0	1	+	–	–	–	–	–	+	–	+	+	–	–	–	+	–	–

A “+” indicates that the problem was solved by the corresponding subject, while a “–” indicates that this was not the case.

TABLE 7
Possible States and Response Patterns for Experiment II

Knowl. space	*Number of states*	*Number of congruent patterns*	*Number of non congruent patterns*	*Number of different congruent patterns*	*Number of different non congruent patterns*
$\mathcal{K}_1$	85	7	39	4	36
$\mathcal{K}_1'$	35	6	40	3	37
$\mathcal{K}_2$	575	23	23	18	22
$\mathcal{K}_2'$	213	20	26	16	24
$\mathcal{K}_3$	1025	25	21	20	20
$\mathcal{K}_3'$	368	20	26	16	24

TABLE 8
Frequency of Distances Between Response Patterns and Closest States for Experiment II

Knowl.			*Distances*			*Average*
space	*0*	*1*	*2*	*3*	*4*	*Distance*
$\mathcal{K}_1$	7	18	13	7	1	1.50
$\mathcal{K}_1'$	6	15	14	10	1	1.67
$\mathcal{K}_2$	23	19	2	1	1	0.65
$\mathcal{K}_2'$	20	16	7	2	1	0.87
$\mathcal{K}_3$	25	17	3	1	0	0.57
$\mathcal{K}_3'$	20	16	7	3	0	0.85

Number of subjects who solved all problems: 4
Number of subjects who solved no problem: 0

"sequence inclusion", whereas it seems to be impossible to decide on the basis of these results between the last two principles.

Examine in Table 8 the observed symmetric distances between response patterns and closest states and compare the number of states with the number of response patterns congruent with states and response patterns not congruent with states. This examination shows that the principles "multiset inclusion" and "sequence inclusion" seems to be adequate for the ordering of tactical chess problems.

The additional assumption concerning the motive "fork" leads, as in Experiment I, to a reduction of congruent response patterns and to an increase in the average symmetric distances. Because the quasi-ordinal knowledge spaces $\mathcal{K}_2'$ and $\mathcal{K}_3'$ derived from this assumption are contained in the corresponding spaces, $\mathcal{K}_2$ and $\mathcal{K}_3$ we cannot conclude from this increase of empirical violations that the additional assumption concerning the motive “fork” is inadequate. We refer to this problem in the general discussion below.

GENERAL DISCUSSION

In the discussion of our experiments we already mentioned that it is impossible to compare the quality of our quasi-ordinal knowledge spaces using only the observed number of congruent patterns and the observed average symmetric distance between response patterns and spaces. The reason is that these numbers depend on the size of a space. For smaller spaces we will expect more violations than for larger spaces. In the following we will suggest three approaches to solve this problem. These approaches try to compare spaces of different size by relativizing the number of observed violations to the size of the space[10].

First, we discuss an approach of van Leeuwe (1974). This approach evaluates the fit of a surmise relation $\preceq$ to a binary data matrix by comparing the observed correlations between items with the expected correlation if it is assumed that $\preceq$ is correct. The fit between surmise relation and data is measured by the so called *correlational agreement coefficient* CA, which is defined by

$$\mathrm{CA} := 1 - \frac{2}{m(m-1)} \sum_{i<j} (r_{ij} - r^*_{ij})^2 \ ,$$

where m is the number of items, r_{ij} is the Pearson–correlation (Phi-coefficient) between item i and item j, and r^*_{ij} is defined by

$$r^*_{ij} \ = \begin{cases} 1 & \text{if } i \preceq j \wedge j \preceq i \\ \sqrt{(1-p_i)p_j/(1-p_j)p_i} & \text{if } i \preceq j \wedge j \not\preceq i \\ \sqrt{(1-p_j)p_i/(1-p_i)p_j} & \text{if } i \not\preceq j \wedge j \preceq i \\ 0 & \text{otherwise} \end{cases}$$

Here p_i is the proportion of subjects that solved item i and p_j the proportion of subjects that solved item j. The higher the value of CA is, the better is the fit between surmise relation and data. For a detailed description of this approach see van Leeuwe (1974).

A second approach for the comparison of surmise relations concerning their fit to a given data set is to count for each relational dependency $x \preceq y$ in a surmise relation $\preceq$ how often it is violated. A violation means here that a subject solved x and failed in solving y. The fit between a surmise relation $\preceq$ and a binary data

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[10] We have to mention here that all three measures are pragmatical approaches to compare the fit of spaces to observed data, which are theoretically not well founded. Models, which relate space size and random influences (lucky guesses or careless errors) to the number of observed violations of a space are, at the moment, not available.

matrix is then measured by the *violational coefficient* VC defined by,

$$\text{VC} := \frac{1}{n(|\preceq| - m)} \sum_{i,j} v_{ij} ,$$

where n denotes the number of subjects, m the number of problems, $|\preceq|$ the number of relational dependencies in $\preceq$ and v_{ij} is the number of violations of the relational dependency $i \preceq j$ if this dependency is contained in $\preceq$ and 0 otherwise. We use $(|\preceq| - m)$ in the denominator of the fraction, because the m relational dependencies of the form $i \preceq i$ can not be violated empirically. VC can be interpreted as the average number of violations of relational dependencies $i \preceq j$ with $(i \neq j)$ contained in $\preceq$. Therefore the lower the value of VC is, the better is the fit of $\preceq$ to the data.

A third approach, described in Schrepp (1993), makes use of the average symmetric distance between a knowledge structure and a binary data matrix. The fit between a knowledge structure $\mathcal{K}$ and the data matrix is measured by the so-called *distance agreement coefficient* DA defined as,

$$\text{DA} := \frac{\text{ddat}}{\text{dpot}} ,$$

where ddat is the observed average minimal symmetric distance between $\mathcal{K}$ and the observed set of response patterns and dpot is the average minimal symmetric distance between $\mathcal{K}$ and the power set of the item set[11]. The value dpot can be interpreted as the expected average symmetric distance if $\mathcal{K}$ contains no informations concerning the solving behavior of subjects, in other words if the data are randomly chosen. The value of dpot decreases with the size of $\mathcal{K}$. Hence, a decrease of ddat with an increase of the size of $\mathcal{K}$ is compensated by an increase of dpot. A lower value of DA indicates a better fit of $\mathcal{K}$ to the data. For more details concerning this approach, see Schrepp (1993).

Table 9 shows for each of our surmise relations $\preceq_1$,$\preceq_1'$,$\preceq_2$, $\preceq_2'$, $\preceq_3$, and $\preceq_3'$, respectively the corresponding quasi-ordinal knowledge spaces, the values of the coefficients CA, VC, and DA. Because there are no systematical differences between the data from Experiment I and Experiment II, we have calculated the values of the coefficients[12] for the union of the data sets.

The three measures, although they are based on different ideas of relativizing the number of observed violations to the size of the space, coincide perfectly in the assertion that the principle "set inclusion" is less adequate for the description of the difficulty of chess problems than the principles "multiset inclusion" and

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[11] Formally we have ddat $:= \sum_{P \in \mathcal{D}} \min(\{d(P,K) \mid K \in \mathcal{K}\}) / \mid \mathcal{D} \mid$, where $\mathcal{D}$ denotes the set of all observed response patterns and dpot $:= \sum_{P \in \mathcal{P}} \min(\{d(P,K) \mid K \in \mathcal{K}\}) / \mid \mathcal{P} \mid$, where $\mathcal{P}$ denotes the power set of the item set.

[12] Remember that a higher value of CA indicates a better fit of a space to the data, whereas for VC and DA the converse holds.

TABLE 9
Values of the Coefficients CA, VC and DA for Our Data From Experiment I and Experiment II

Knowl. space	*CA*	*VC*	*DA*
$\mathcal{K}_1$	0.831	0.057	0.34
$\mathcal{K}'_1$	0.863	0.054	0.34
$\mathcal{K}_2$	0.842	0.025	0.18
$\mathcal{K}'_2$	0.863	0.027	0.21
$\mathcal{K}_3$	0.836	0.024	0.17
$\mathcal{K}'_3$	0.867	0.027	0.21

"sequence inclusion". On the basis of our results, it is impossible to decide between the last two principles, because the values of all three coefficients are nearly identical for the spaces $\mathcal{K}_2$ and $\mathcal{K}_3$.

Remember that the relations $\preceq'_2$ and $\preceq'_3$ differ only with respect to the relational dependencies $(G, E, S) \preceq'_2 (E, G, S)$, $(E, G, S) \preceq'_2 (G, E, S)$ and $(G, E, S) \preceq'_2 (E, E, G, S)$, which do not hold for $\preceq'_3$.

The relational dependency $(G, E, S) \preceq'_2 (E, G, S)$ is violated by 10 of our 92 subjects: 10 subjects solved (E, G, S) and failed in solving (G, E, S). The relational dependency $(E, G, S) \preceq'_2 (G, E, S)$ is violated by 3 subjects, whereas $(G, E, S) \preceq'_2 (E, E, G, S)$ is violated by 14 subjects. So at least the two relational dependencies $(G, E, S) \preceq'_2 (E, G, S)$, and $(G, E, S) \preceq'_2 (E, E, G, S)$ are not supported by our data.

The spaces $\mathcal{K}'_2$ and $\mathcal{K}'_3$, which are based on the additional assumption that the motive "fork" can be recognized more easily than the other motives in our motive set, show accordingly to the DA and VC coefficients a lower fit to the data than the spaces $\mathcal{K}_2$ and $\mathcal{K}_3$, whereas for the CA coefficient the converse holds. Therefore, no clear assertion concerning the adequacy of this additional assumption can be drawn. The surmise relations $\preceq'_2$ and $\preceq'_3$ based on this additional assumption show a better approximation to the observed correlations between our chess problems than the relations $\preceq_2$ and $\preceq_3$. However, they also show a worse approximation concerning the violation of relational dependencies contained in them and observed minimal symmetric distances relativized to the size of the corresponding spaces.

Another point that should be discussed is the suitability of motives for the characterization of chess problems. Motives may only be adequate as "exclusive" components of simple problems as used in our investigations. However, other features of the problems may influence their difficulty. For example, the difficulty of a problem may also be dependent on the number of pieces that appear on the board. However, a surmise relation which is solely based on the number of pieces, for example, leads to an average symmetric distance of 2.74 in Experiment I and of 2.63 in Experiment II, which are extremely high compared to the average symmetric distances of our best space $\mathcal{K}_3$ (0.5 and 0.57 in Experiment I and Experiment II) respectively. In our experiments, we tried to minimize the influence of other problem elements by striving for maximal homogeneity of the problem set. With more complex problems, the set of relevant problem components will still contain motives, but other factors that are indispensable in the knowledge of a good chess player may become relatively more important. More information concerning these other factors can be gained from the investigations of De Groot (1965). However, it seems possible to capture also these factors by the theory of knowledge spaces or one of its generalizations.

Our investigation was based on an extension of the theory of Doignon and Falmagne (1985, 1998). We have provided principles for the establishment of surmise relations and quasi-ordinal knowledge spaces that are primarily dependent on problem components. In addition to Doignon and Falmagne, who do not consider the properties of the problems themselves, we are particularly interested in those components of problems that may be fundamental to the surmise relation and the corresponding quasi-ordinal knowledge space. In both our theoretical and empirical investigations we were able to show that the theory of knowledge spaces can be used for testing psychological assumptions, which may be fruitful for the theory of problem solving in general.

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